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# Proposition for a new mechanics for fluid turbulence by the theory of scale relativity (SR) (continued): Application to rotating turbulence

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# Plan of exposition (1)

- 1) Introduction
- 2) a glimpse at scale relativity theory (SR) and conditions for its application
- 3) application of SR to turbulence
- 3.1 reminder of previous results in homogenous isotropic turbulence (HIT)
- 3.2 application of SR to rotating turbulence
- (in course)
- Conclusions (provisory)

# 1. Introduction

- Fluid Turbulence : NS et Euler equations, still puzzling since the nature of possible singularities in (x,t) if any are not yet known…(singularities are thought to be central for intermittency /large acceleration tails)

Multi-fractal behavior of variables well observed in data but the theory is still at the level phenomenology/not solving the true dynamics.

- Proposal here for a new approach (less phenomenological? ) based on a macro-quantic Schrodinger equation. Link with NS-Euler eqs through the covariant derivative in the stochastic case , reflecting fractal geometry.

# 1) Fluid turbulence (1)

- Fluid described by NS , Euler or Burgers equations (p=0); MHD case (with B in addition) Quadratic Non Linearities (see equations) non local one for pressure p when it exists.
- Ex of open pbs : in 3D
- a) are these NS , Euler, Burgers equations still valid in the turbulent regime (possible new terms, expansion in grad(v) …)
- b) if yes, the nature of their singularities is not yet known…
- c) passage NS->Euler (zero viscosity) still not well understood (maths: finite blow up or not and so on …) ; ex : energy cascades are observed but not yet suitably predicted (example : phenomenology of multiplicative cascades…)
- d) intermittence (definition) still to understand
- Caracteristics of turbulent field v (and p), and vorticity fields …: non differentiable, bundle of paths going at “ random”….

# 2. Fluid turbulence and SR

- Kolmogorov phenomenology scale invariance (but it is not exactly like this in real experiments)
- 3 main ingredients :I+II+II to apply the Scale relativity (LN), theory with underlying fractal geometry ->scale dependence in x,t of the involved variables , but also in dx, dt . Here we adopt a Lagrangian approach. SR->QM in the x space
- I) chaotic -> stochastic description ; Brownian motion but now in v space and « K41 law » , with fractal dimension $D_f=2$ : $\delta v^2 \approx (Dsct)\, \delta t$ (more usual in x : $\delta x^2 \approx \delta t$ )
- II) existence of an infinity of trajectories (here of Lagrangian tracer in v space)
- III) local irreversibility (symmetry breaking) in dt->-dt
- (different from stochastic mechanics for which t->-t breaking)
- ->two-valuedness of acceleration…(more usual of velocity)

- Additional conditions for application of SR to turbulence :
- 4) Suppose a number of scales sufficient in particular in the inertial range /Re sufficiently high (ratio $TL/\tau_\eta$ of integral time versus K dissipation time : $TL/\tau_\eta = Re\lambda/2C_0$ , $C_0 = 4$ το 7 , see good number is of order 100 or more)
- 5) work on an equation such as the Newton equation (as with. the NS equation)
- cf other approach :like Stochastic NS, with proper choice of the forcing source term (in x and in t) …Generally we find a Schrodinger operator by applying a covariant like derivative when starting with Newton or NS-Euler eqs.

# 2. SR and covariant derivative

- The supposed underlying stochastic background (in x,t) is encoded in the covariant derivative D^ /Here the turbulent velocity field itself induces a fractal (scale dependent) non differentiable such a backgound)
- D^ expresses the fractal geometry (like the Einstein covariant derivative for curved space-time)
- D^ allows to go from NS-Euler eq. to Schrodinger eq. (in x space in QM, in v space here)

# Application to turbulence :

# *v* = is now the basic coordinate

$$dv = a\,dt + dW \qquad dW = \zeta\sqrt{2D_v dt}$$ (K41)

$$a \longrightarrow \{a_+, a_-\} \quad \rightarrow \mathcal{A} = \frac{a_+ + a_-}{2} - i\,\frac{a_+ - a_-}{2}$$

(Irreversibility -> doubling of acceleration vector)

$$\mathcal{A} = -2iD_v\nabla_v \ln\psi_v$$

(potential part of acceleration)

$$\frac{\widehat{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{A}.\nabla_v - iD_v\Delta_v$$

(New form of total derivative)

→ Motion equation : Navier-Stokes ▭ ($\rho$=1,if incompressible)

$$\frac{dv}{dt} = F = -\nabla p + \nu\Delta v$$

Derivative of NS: $$\frac{da}{dt} = \dot{F}$$ ▭

$$\frac{\widehat{d}}{dt}\mathcal{A} = \dot{F}$$

# Schrödinger form of mean motion equation in special case : $dF(v,t)/dt=-grad_v(\Phi(v,t))$

$$D_v^2 \, \Delta_v \psi_v + i \, D_v \frac{\partial \psi_v}{\partial t} - \frac{\phi}{2} \psi_v = 0$$

$$P_v(v) = |\psi_v|^2$$

With dissipation -> Non-Linear Schrödinger , with a term in ψ ln ψ → **Quantized**

Back to stochastic initial description : ***A* is now known**

$$\psi_v = \sqrt{P_v} \times e^{i\theta} \qquad dv_+ = A_+ \, dt + dW$$

$$A_+ = A_U + A_V = D_v(\partial_v \ln P_v + 2\partial_v \theta)$$

Sub-special case : $\theta$ = cst (ex 1D . : harmonic oscillator, psi *real*) --> $A_+ = A_q$

$$A_q = D_v \, \partial_v \ln P_v$$

Refs. : L. Nottale, arXiv:1306.4311; Cells 2014, 1, 1-35; LN, *Space-Time Geometry and Quantum Events*, Chapter 5, Ed. I. Licata, Nova Science Pub., 2014. LN+TL , POF 2019

# 3) Application of SR to fluid turbulence

- 3.1 previous results of SR applied to HTI : a selection …
- 3.2 rotating turbulence
- Hope : Looking for recovering previous results of HTI in this case but now expectating also getting new features : predictions and verifications to be performed on relevant available data.

# 3.1 previous results of SR applied to HTI (isotropic case)

- Main predictions and results /+figures
- Looking for classical and quantum oscillators
- Pdf(v)/Pdf(a)
- Comparison with experiments

# Reminder of main results of SR in (HIT) turbulence

relativity methods in *velocity*-space to turbulence description

(réf : L. De Montera, 2013, **A theory of turbulence based on scale relativity,** arXiv:1303.3266)

- New *Schrödinger-like* form of motion equations (on the time derivative of NS eqs.) in v space
- Main consequence: existence of *a new* « divergent » *component of acceleration*
- Comparison to global experimental data : *aim to describe large tails* of the accelerations' PDF
- Comparison to local experimental data : a new proposed mechanism for *intermittence* bursts
- (Not shown here : Scaling laws and link to related multifractal studies, see POF 2019)…..

# 3.1 SR/HIT theoretical predictions

- Velocity PDF : globally close to Gaussian (Mordant 2001, Voth et al 2002, etc.), but :
- Locally PDF (v) : strongly deviates from Gaussianity (Heppe 1998, etc.)
- Here: $P_v(v) = |\psi_v|^2$ ; psi > 0 and < 0 → $P_v$ has
- minima at $P_v = 0$

$$A(v) = \mathcal{D}_v \, \frac{\nabla_v P_v}{P_v}.$$

→ Divergence of acceleration on minimas of Pv

= our singularities here

→ **Predicts large tails of acceleration PDF**

$$\hbar_v = 2D_v$$

# Tails of acceleration PDF : ***P*[*v*]=0**

- General case : near a zero of Ψ: $P_v \propto (v - v_1)^2$
- Corresponding acceleration (for $v_1 = 0$):

$$P_v(v) = \left(\frac{v}{v_0}\right)^2 \qquad A(v) = 2\mathcal{D}_v/v$$

- Derived asymptotic PDF of acceleration P(a) from P(v) by inversion :

$$P_a(a) = \sum_k \frac{1}{|A'[V_k(a)]|} P_v[V_k(a)],$$

$$\boxed{P_a(a) = \frac{(2\mathcal{D}_v)^3}{v_0^2} \frac{1}{a^4}}$$

# General method to go from PDf(v) to PDF(a) (1)

- To get the PDF Pa[a]: we now ψ[v] then P[v] = /ψ[v]/$^2$
  We make a realization of P[v] with vi distributed according to P[v]; we know Aq[v] = Dv ($\partial_v$ P[v]/ P[v]) ; we compute the values Aq[$v_i$] for each $v_i$
  Then we an histogram of these Aq[$v_i$] which yields the seaked PDF : Pa[a];

- For functions :
- We call V(a)=$A^{-1}$(a) the inverse of the function a=A(v), then we use the following inversion formula :

# General method to go from PDf(v) to PDF(a) (2)

these parts. The resulting probability distribution of acceleration is given by the inversion formula :

$$P_a(a) = \sum_k \frac{1}{|A'[V_k(a)]|} P_v[V_k(a)], \qquad (2)$$

where the sum is done on each of the monotonic parts of the inverse function $V = A^{-1}$.

In the simple case when there only one part, this formula becomes :

$$P_a(a) = \frac{P_v[V(a)]}{|A'[V(a)]|}. \qquad (3)$$

# General method to go from PDf(v) to PDF(a) (3)

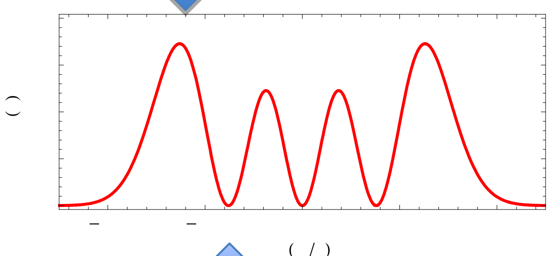
- Applications :
- For a Gaussian PDF in v: P(v) in $\exp(-(v/\sigma_v)^2)$
- We get a $P_a(a)$ in $\exp(-(a/\sigma_a)^2)$, with $\sigma_a = D_v/\sigma_v$
- ($D_v$ is in v.a) : Global case ->no intermittency
- Need to look at local pdf 's ...
- Application to harmonic oscillator (HO) , we get the squared Lorentzian for P(a) , by using he inversion formula for explicit solution of the HO, for P(v). See examples next...

# Test of theoretical predictions with experimental data : acceleration PDF

Examples: quantized harmonic oscillator,
≈ particle in a *v*-box, etc.. -->

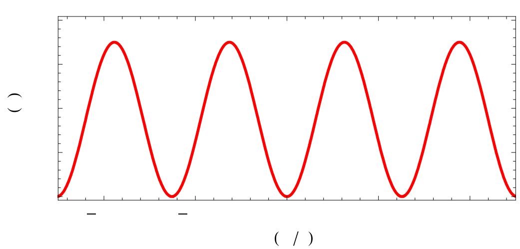
$$P_a(a) = \frac{2}{\pi\sigma_a}\,\frac{1}{[1+(a/\sigma_a)^2]^2}.$$

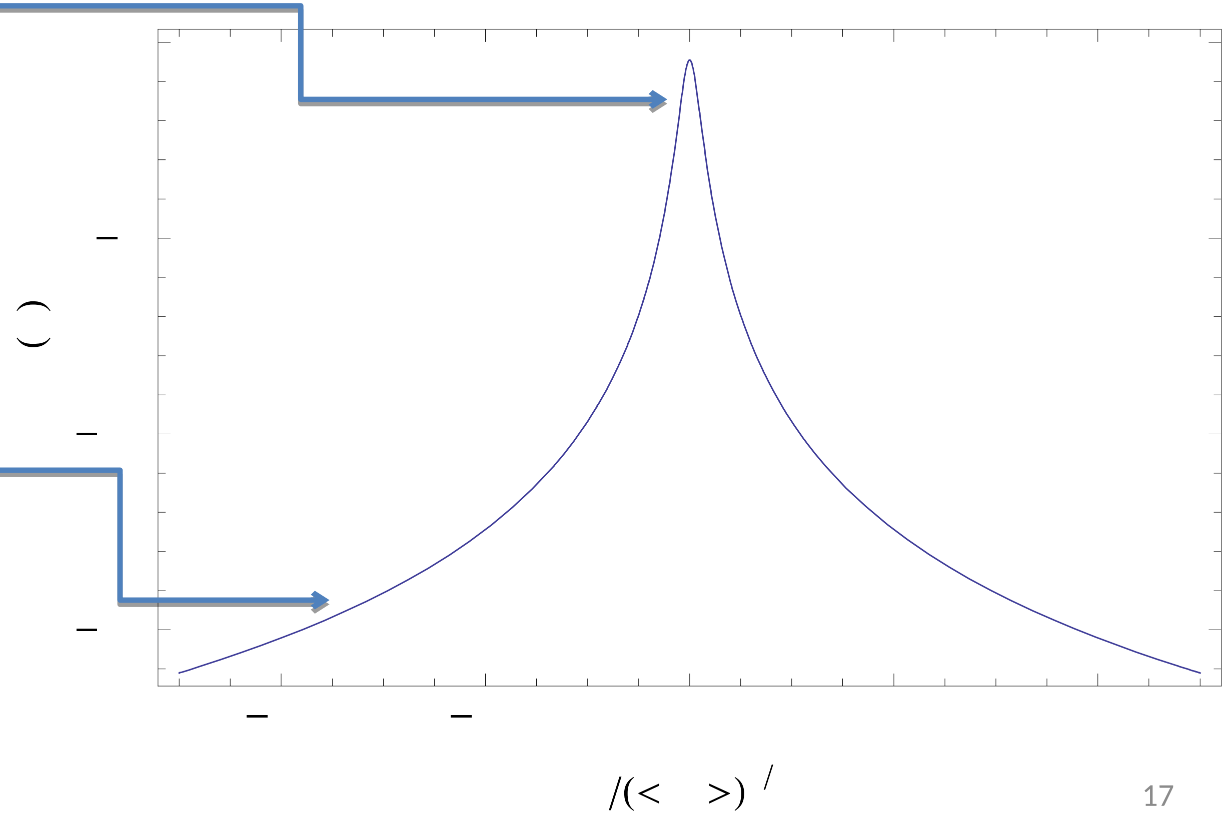
Peak of proba



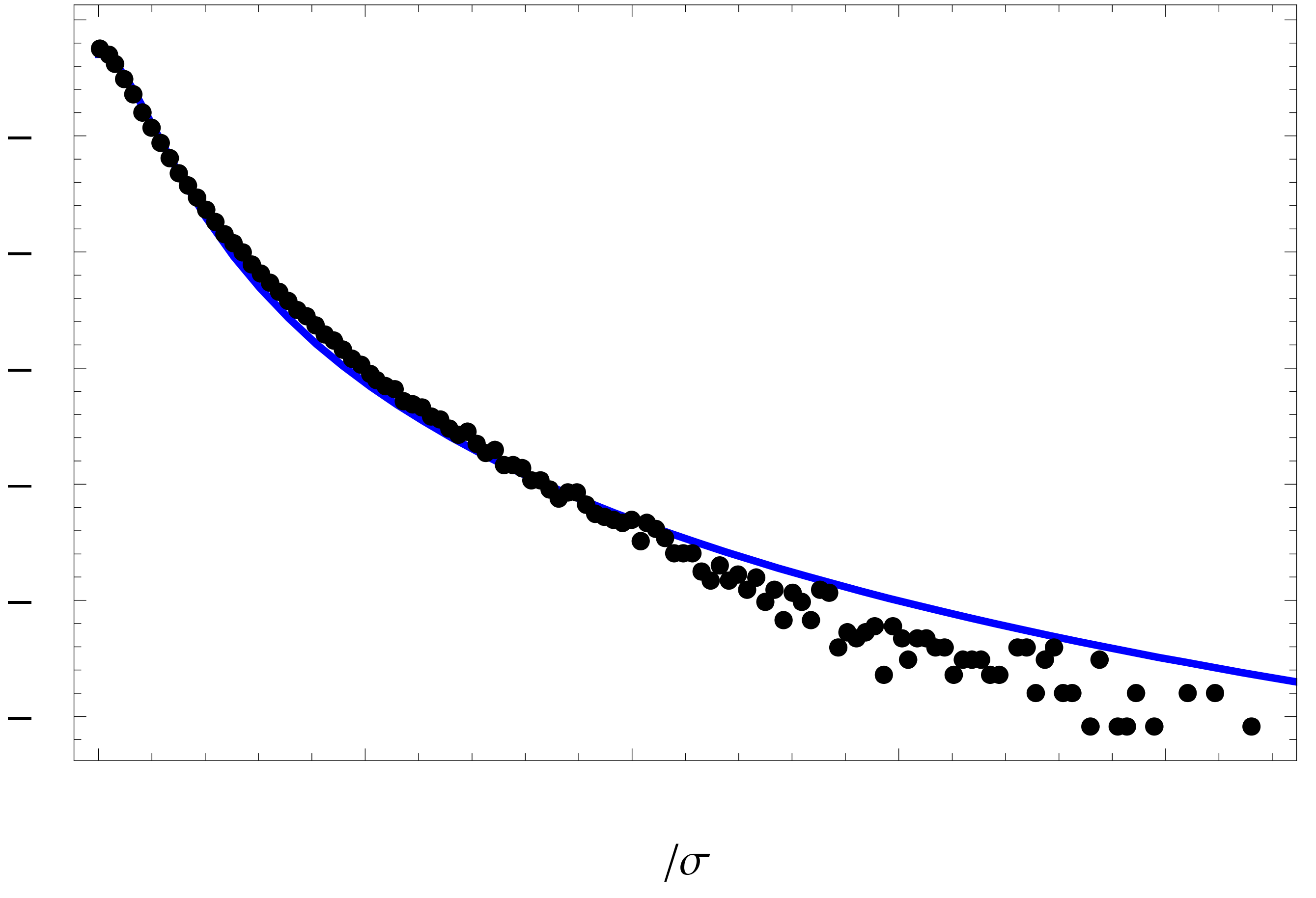
Tails of proba

*v*-Box :





# Comparison with experimental data L (but surestimate the large a)




N. Mordant 2001 PhD thesis. Mordant et al PRL 2001.
Double von Karman flow; Lagrangian tracers 250 μm ; ultrasonic doppler tracking; sampling 6.5 kHz == 0.7 $\tau_\eta$ (0.22ms); $R_\lambda$ = 810; fully developed turbulent flow.

# Pdf(v) for classical or quantum oscillators

- Comparison between the PDFs of a quantized harmonic oscillator in v-space (red curve) to that of a classical harmonic oscillator having the same parameters ($n = 3$, $v_m = 0$ and $v_0 = 0.3$ m/s.) see next : expression and figure.

# Analytical expression of the Pv for harmonic oscillator

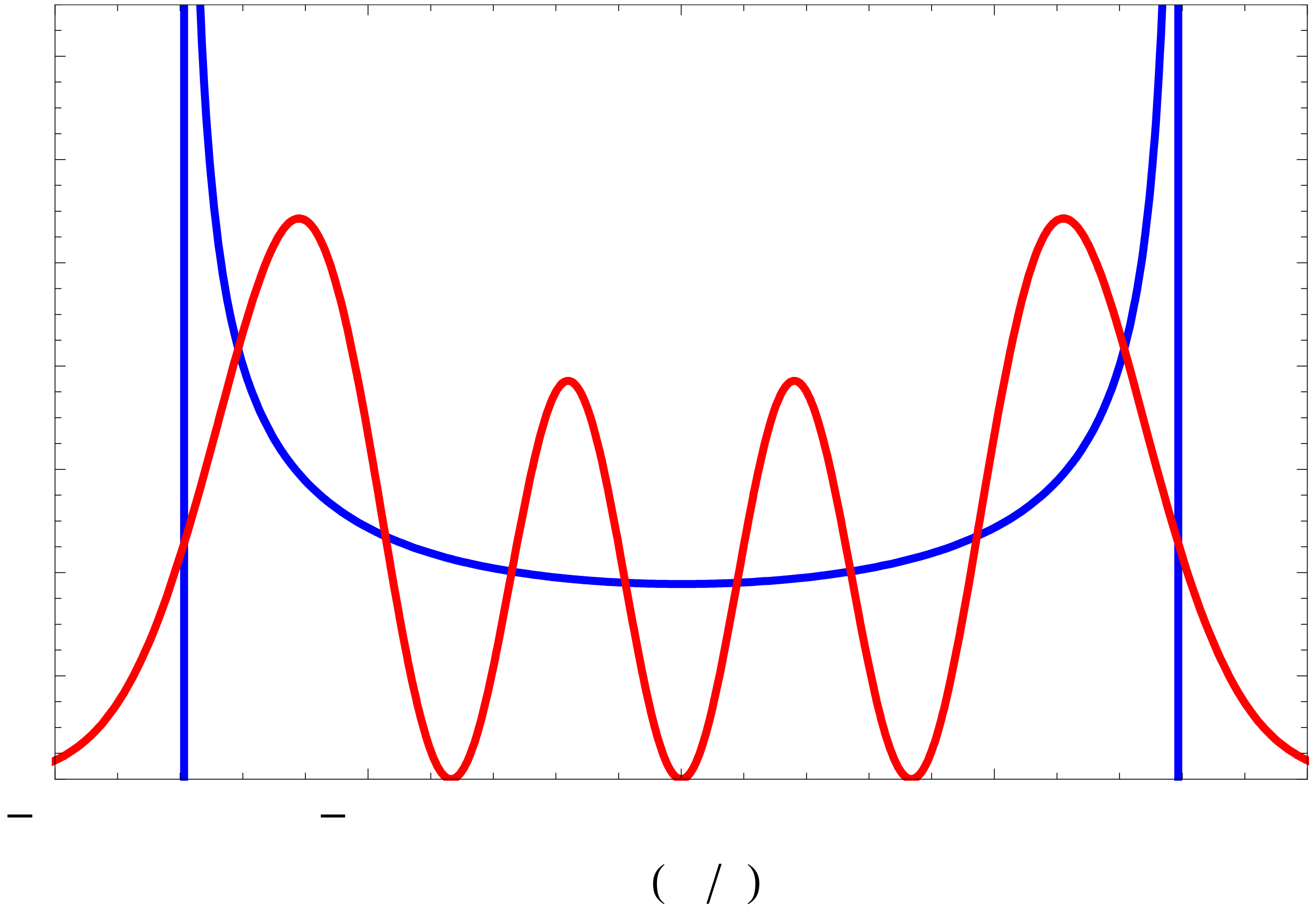
$$P_v[n,v] = \frac{1}{n!\,2^n\sqrt{\pi}}\, H\left[n, \frac{v-v_m}{v_q}\right]^2 \exp\left[-\left(\frac{v-v_m}{v_q}\right)^2\right],$$

$n, v]$ are the Hermite polynomials) compared to that of a classical HO,

$$P_{\rm cl}[n,v] = \frac{1}{\pi\sqrt{2n+1-(v-v_m)^2/v_q^2}}.$$

**To get the PDF(a) from PDFv)=Pv (see other slide)**

# Classical (blue) vs. quantum harmonic oscillator (in red): velocity PDF



n=3; $v_0$ = 0.3 m/s

# Transition classical-’quantum like’ in (a,v) phase space

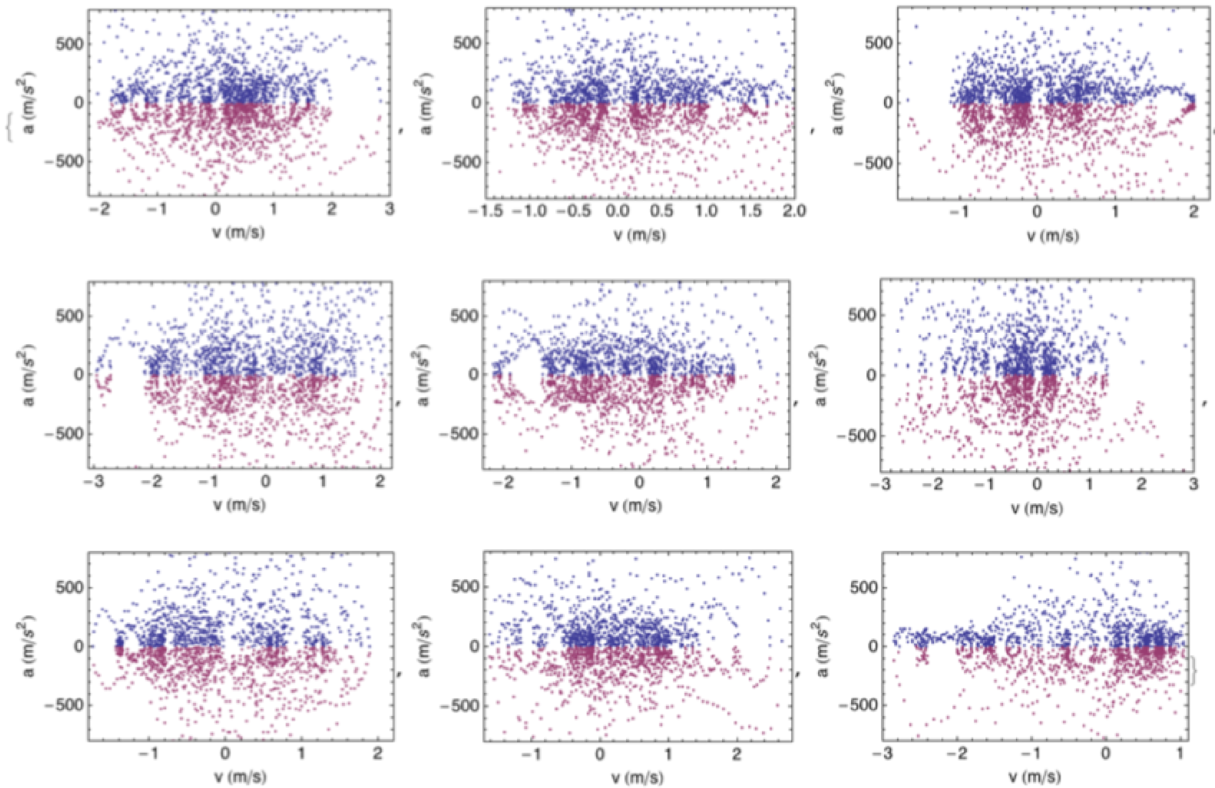
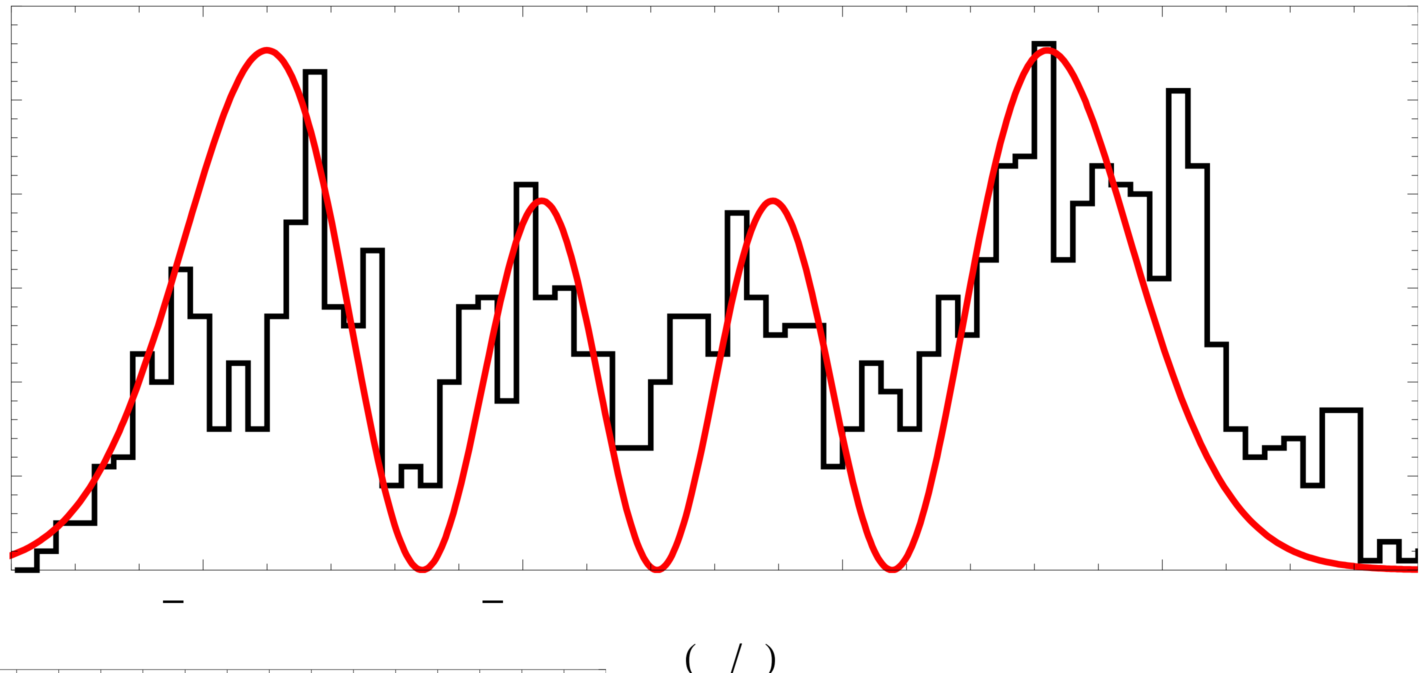


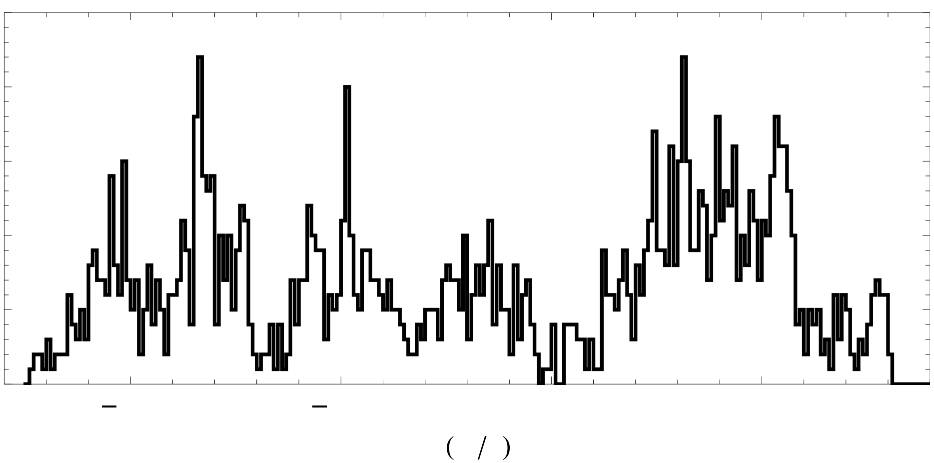

**FIG. 10**. Examples of “phase” diagram ($v$, $a$) for nine long segments of Mordant’s experiment man290501 (respectively, segments 1135, 2578, 3030, 85, 3439, 8292, 9508, 6353, and 3500, whose length vary from 2218 to 1729 $t_s$, i.e., ≈1700 to 1300 $\tau_\eta$ and ≈15 to 12 $T_l$). One can easily check that the existence of almost empty minima in the velocity distribution for $|a| < \approx\sigma_a$ = 280 m/s$^2$ is a systematic property of these Lagrangian segments although the position of these minima varies, as expected, from one segment to the other. On the contrary, the velocity PDFs for $|a| > \approx \sigma_a$ remains smooth and close to Gaussian (see Fig. 13).

# Local quantized harmonic v-oscillator Φ(v) shape on a given real data

Seg3398-1-1770: n=3, $v_0$=0.3

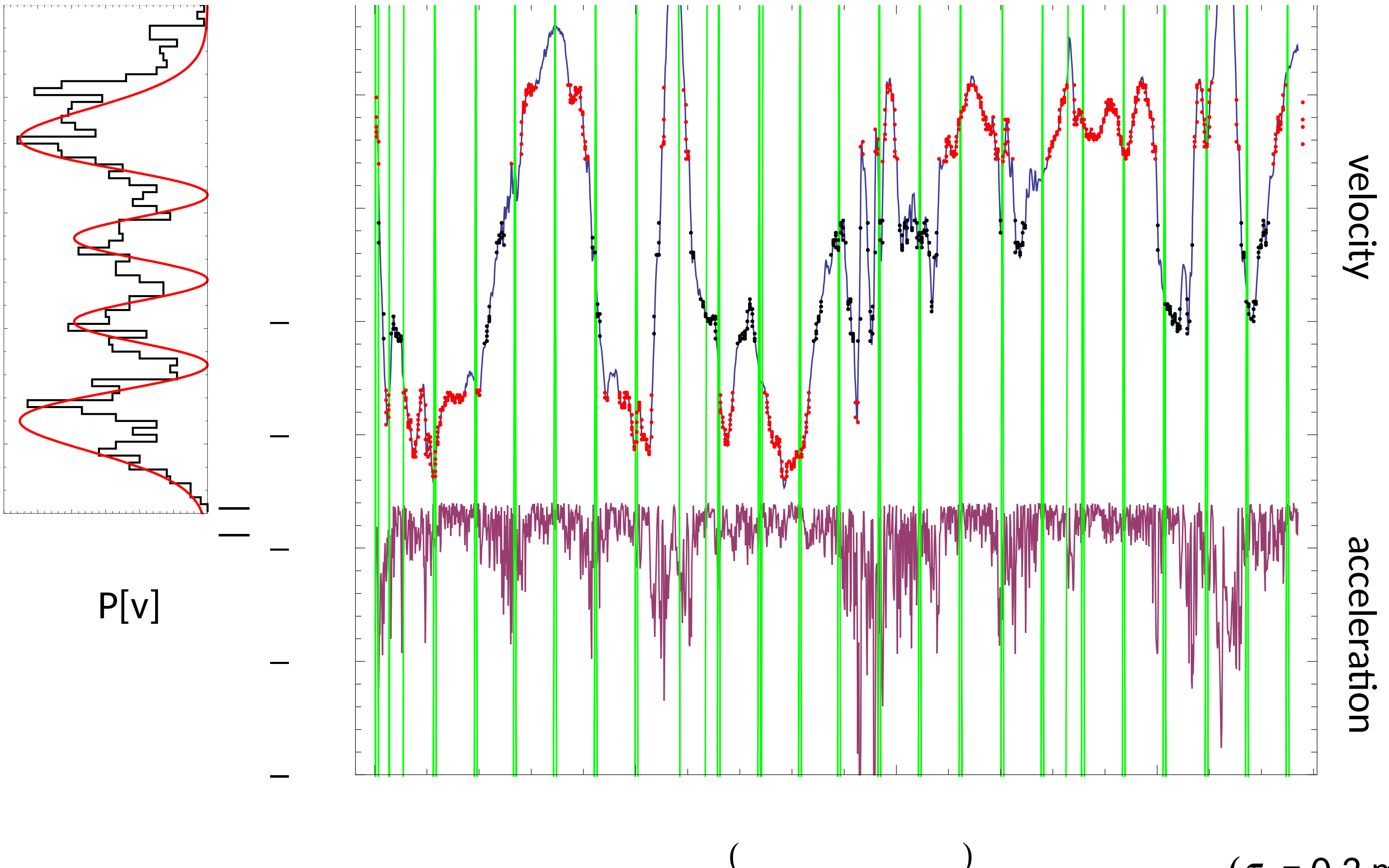
All seg ->





≈ Account of Transition cl-qu $|a| < \sigma_a$

# Mechanics of acceleration intermittence

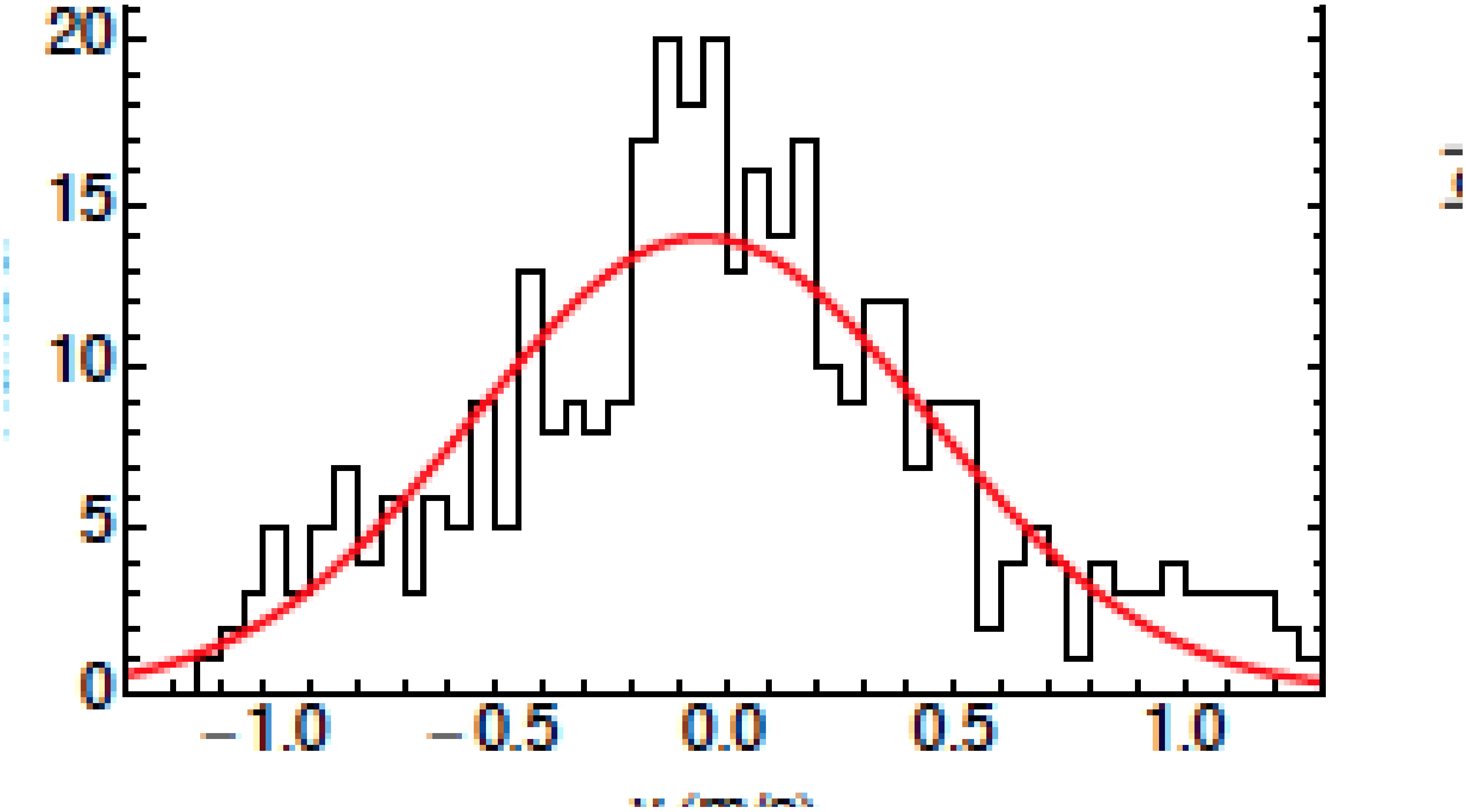



($\tau_\eta$ = 0.2 ms)

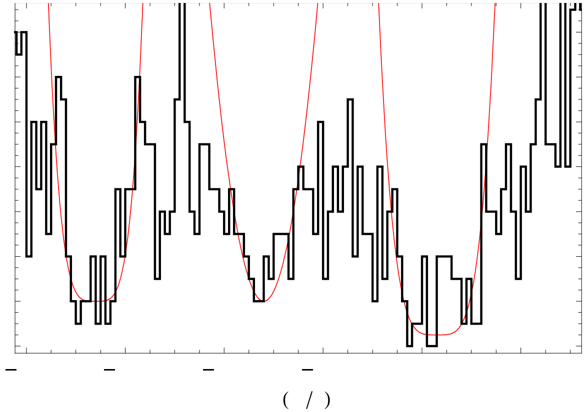
(TL = 22.4 ms = 146 tu)

Quiet periods = particle trapped in main probability peaks
Bursts = macroquantum jumps between probability peaks

# Mechanics of acceleration intermittence, Gaussian shape in classical |a| > $\sigma_a$ zone for P(v)

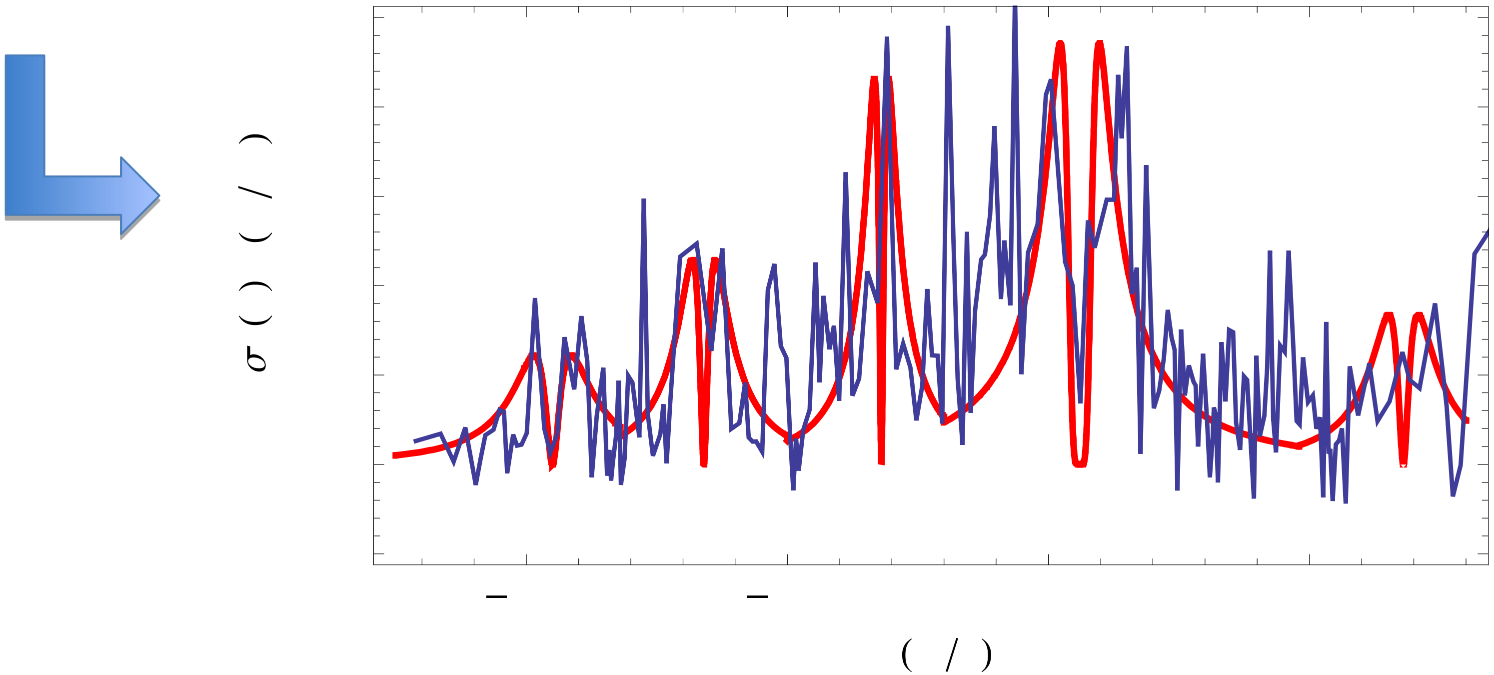


# Modelization of minima of P(v) –> $A_q$(v)

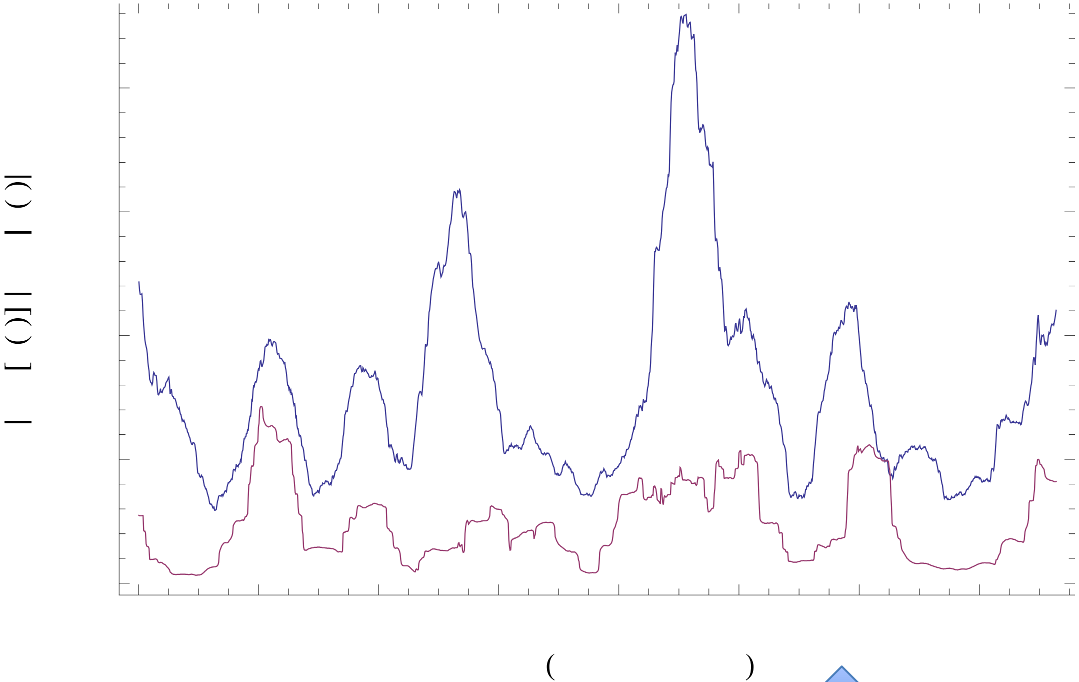


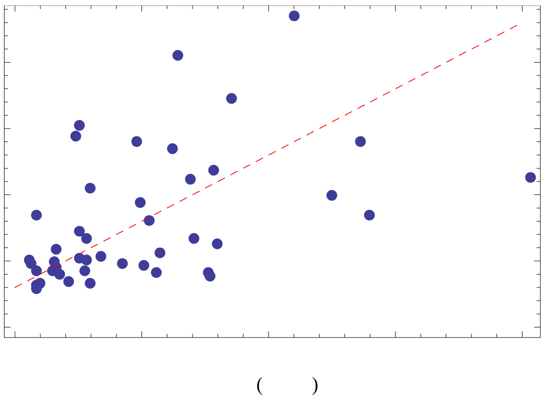
Blue: observed running acceleration dispersion in function of velocity (partition : 9 points)

Red: predicted value of $|A_q|(v)$



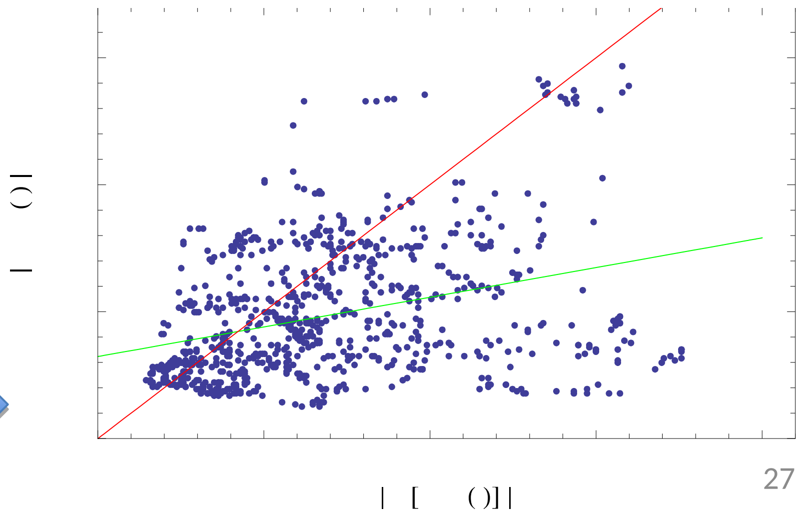
# Predicted $A_q[v(t)]$ vs observed a(t) : best with time averaging





Averaging on $\Delta t = T_L/2 = 73$ tu

Aq[v(t) vs minimas of bursts of a(t).
Statistical significance of correlation :
10.1 sigmas



# 3.2 Application of SR to rotating turbulence

- Aims : examine if the previous results are still valid here (for at least hoped ‘universal’ ones) +get new results
- Predictions of new local PDF(v) and PDF(a) but accounting now for possible anisotropy (on $v_{//}$ and $v_{perp}$ and on $a_{//}$ and $a_{perp}$ (//and perp with respect to the rotation direction)

# 3.2 New prediction : application of SR to rotating turbulence

The NS incompressible equation of motion in the rotating frame reads :

$$D_t u = -\nabla(p + p_c)/\rho - 2\Omega \times u + (\eta \Delta u)$$
$$D_t \rho = -\rho(\nabla . u)(CP = 0)$$
$$D_t = \partial_t + u.\nabla, \; p_c = \rho(\Omega \times r)^2 / 2$$

If again u is fractal field in the sense $du^2$ in dt we can write now a Schrodinger equation in the u space but in presence of a vector potential Ac(u) to account for the Coriolis force such that Curl(Ac)=2Ω x u

# 3.2) SR for rotation (2)

- We can now predict like before another new acceleration
- $A=-2iD_u grad_u(Ln(\Psi(u(t,dt),t))$   (I)
- But now Ψ is solution of a (vectorial) Schrodinger eq. with Ac(u) :

(3) $$(iD_u\partial_t + (iD_u\nabla_u - A_c(u,t))^2 - \Phi_{tot}(u,t)/2)\Psi(u,t) = 0$$

Solutions can be found within 3 D harmonic oscillators : Φ tot=Φ external potential, +Φc centrifugal perpendicular potential (also harmonic) and 3nd potential which comes here from the Ac potential expression.

Solving for stationnary states with HΨ=EΨ, H Hamiltonian is here symmetric in vx,vy variables.

We choose in cartesian coordinates the (gauge) expression Ac=(-Ωy, Ωx,0) for Ω//ez

- exact determination of the eigenfunctions and eigenvalues E(Lz) involving the kinetic momentum $Lz=v_x d/dv_y - v_y d/dv_x$ , prop to ∂ / ∂φ, for eq.(3):

# 3.2) SR for rotation

a) Solving for $v_z$

we get a standard harmonic oscillator solution

assuming an initial potential in $v_z$ as $\omega_z^2 v_z^2/2$

->Pdf($v_z$) in $H_{nz}(\beta_z v_z)^2 \exp(-\beta_z^2(v_z - v_{z0})^2)$ as above in HIT

$2\beta_z^4 Dv_z^2 = \omega_z^2/2$ ; With quantized energy $E(n_z) = 2Dv_z\, \omega_z\, (n_z + 1/2)$

we get the associated Aq as in HTI :

$$A_q(v_z) = \pm \mathcal{D}_{vz}\, \partial_{vz} \ln P_v = \pm\, 2D_{vz} \left( 2n_z \frac{\mathcal{H}_{n_z-1}(\beta_z v_z)}{\mathcal{H}_n(\beta_z v_z)} - \beta_z^2 v_z \right)$$

b) Solving for $v_{perp}$ ($v$, $\varphi$)

The original equation is :

$(Dv^2\Delta v + v^{-2}\partial^2\varphi - 2i\Omega Dv\partial\varphi - (\omega + \Omega)^2 v^2/2 + E))\, G(v, \varphi) = 0$ it can be solved exactly only if we take the

# 3.2 SR for rotation

- the eigenvalues of the $L_z$ ($\partial\varphi$) are in $l_{(z)}$, and those of $L_z^2$ in $l(l+1)$…Thus factorizing by this Ansatz as $G(v, \varphi)=g(v)\, h(\varphi)$ with $h(\varphi)$ in $\exp(\, il\, \varphi\, )$; It remains a radial equation as :
- $$\left(\mathcal{D}_v^2\, \Delta_v - \frac{l_z(l_z+1)}{v^2} + 2\,\Omega\, \mathcal{D}_v l_z - \frac{1}{2}(\omega^2+\Omega^2)v^2 + E\right) g(v) = 0.$$

$$g(v) = c_n v^{l_{1z}}\, \mathcal{H}_n(\beta v)\, e^{-\frac{1}{2}\beta^2 v^2}$$

with $2\beta^4 Dv^2=(\omega^2+\Omega^2)/2$ , $l_{1z}{}^2=l_z\,(l_z+1)$

With quantized energies $En,l=2Dv\,(-\Omega l_z+(\omega^2+\Omega^2)^{1/2}(n+l_{1z}+1))$ ; $E_{n,l,nz}=Ez(nz)+En,l$ ;

PDF(v) in $H_n(\beta\, v)^2\exp(-\,\beta^2(v-\,v_0)^2)$ $v^{2l1z}$ : NEW….

# Pdf(v), a , Pdf(a) in // and perp directions (1)

- $Pdf(v,v_z)=Pdf(v).Pdf(v_z)$ , P could be anisotropic
- Accelerations are given by :
- for az :

$$A_q(v_z) = \pm \mathcal{D}_{vz}\, \partial_{vz} \ln P_v = \pm\, 2D_{vz} \left( 2n_z \frac{\mathcal{H}_{n_z-1}(\beta_z v_z)}{\mathcal{H}_n(\beta_z v_z)} - \beta_z^2 v_z \right)$$

- with same predictions as in HTI for asymptotic
- Pdf(az) in $az^{-4}$ and central pdf(a) with squared Lorentzian shape +exponential cut-off due to dissipative scales (not shown in this talk).

# Pdf(v), a , Pdf(a) in // and perp directions (2)

- For aperp :
- $$A_q(v) = \pm \mathcal{D}_v\, \partial_v \ln P_v = \pm\, 2D_v \left( 2n \frac{\mathcal{H}_{n-1}(\beta v)}{\mathcal{H}_n(\beta v)} + \frac{l_z}{v} - \beta^2 v \right)$$
- with again same predictions as in HTI for asymptotic PDF(aperp) in $a^{-4}$ and central pdf(a) with squared like Lorentzian shape +exponential cut-off due to dissipative scales but also a term in lz/v in addition in Aq(v).

- ->DEFORMATION of the shape of the PDF(aperp), see next…

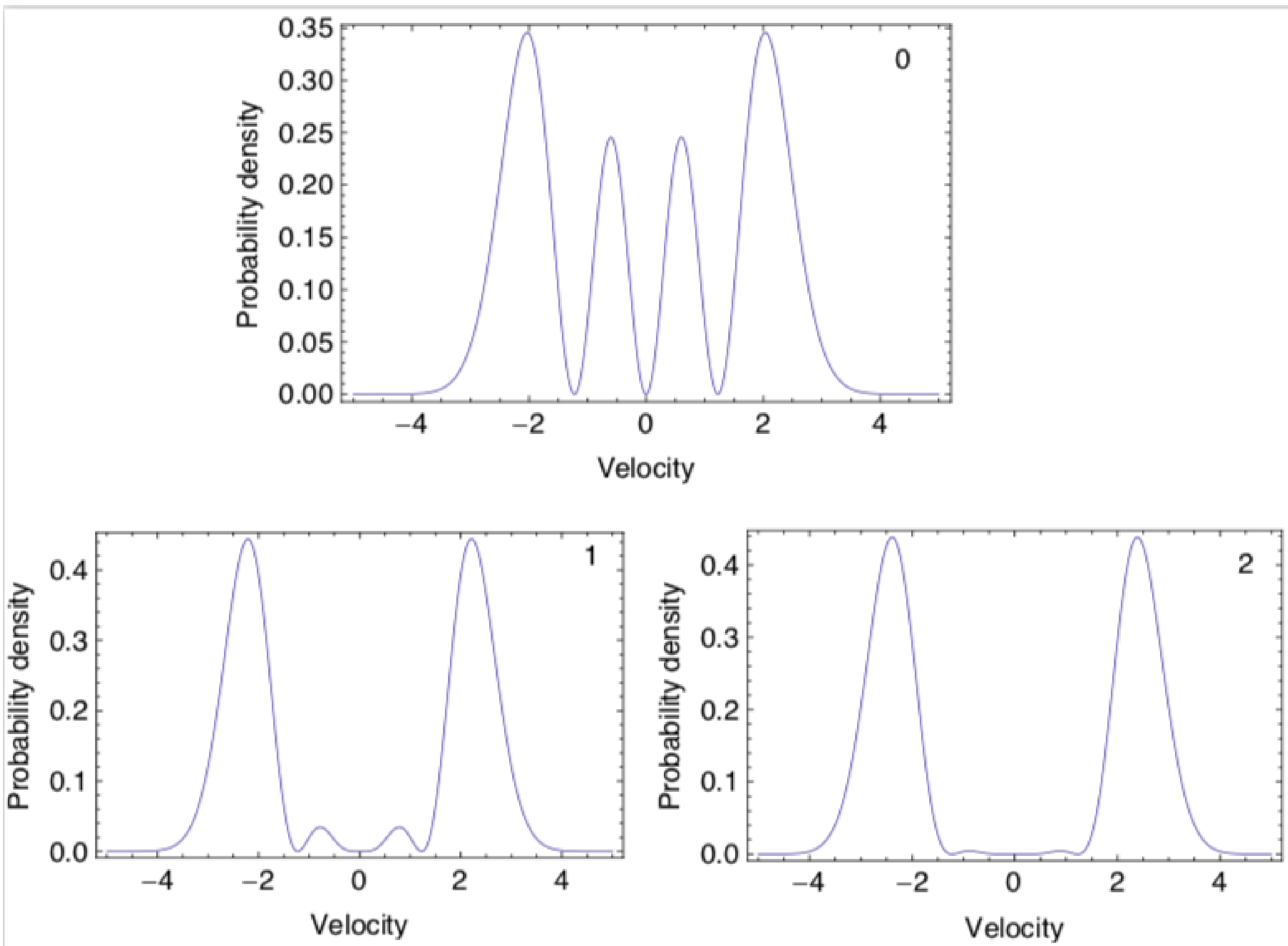
# New predictions in the perp direction

- Thus the $v^{2l1z}$ contribution in Pdf(v) involves a flattening of the central regions (v around 0) in v space; wich in turn leads to an enlargement of the high acceleration zone around v=0 by the lz/v term in Aq(v).
- See next figures

# New prediction for P(v) and Aq(v)

- Figure (1): PDF of velocities generated by the solution ψv of the Schrodinger-Coriolis equation (withPv =|ψv|$^2$,for $v_0$ =1(arbitrary unit),n=3 and lz =0 to2. The value lz =0 corresponds to an absence of Coriolis force (upper left figure). The presence of a Coriolis force involves a suppression of the inner secondary peaks increasing with the value of lz, leaving a large empty central band with almost forbidden velocity values.

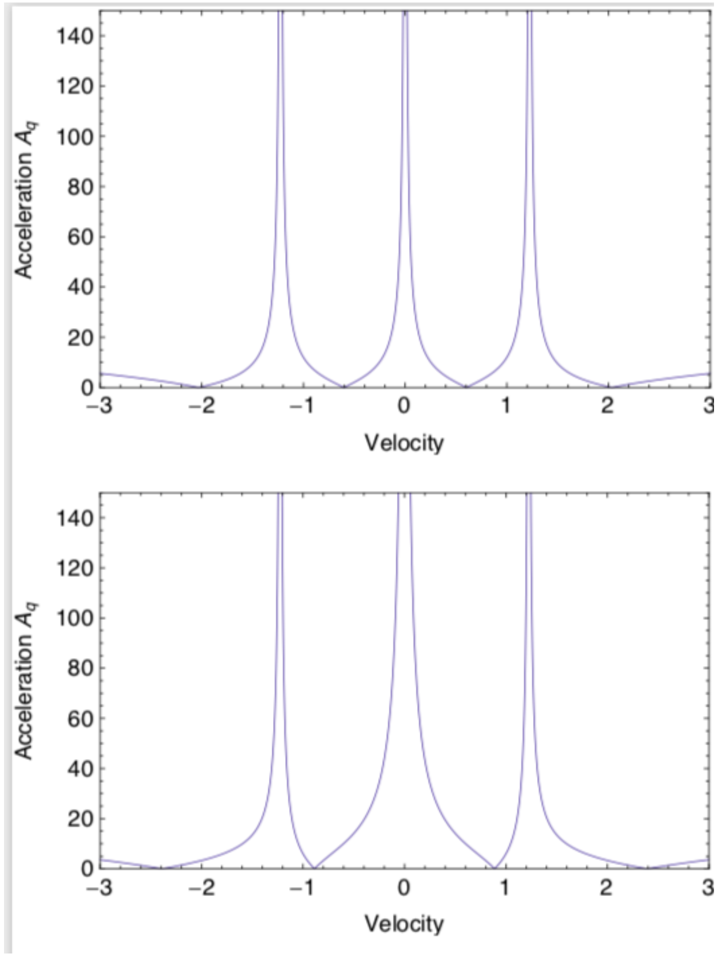
# Figure for Pdf(v) for l=0,1,2 (n=3)



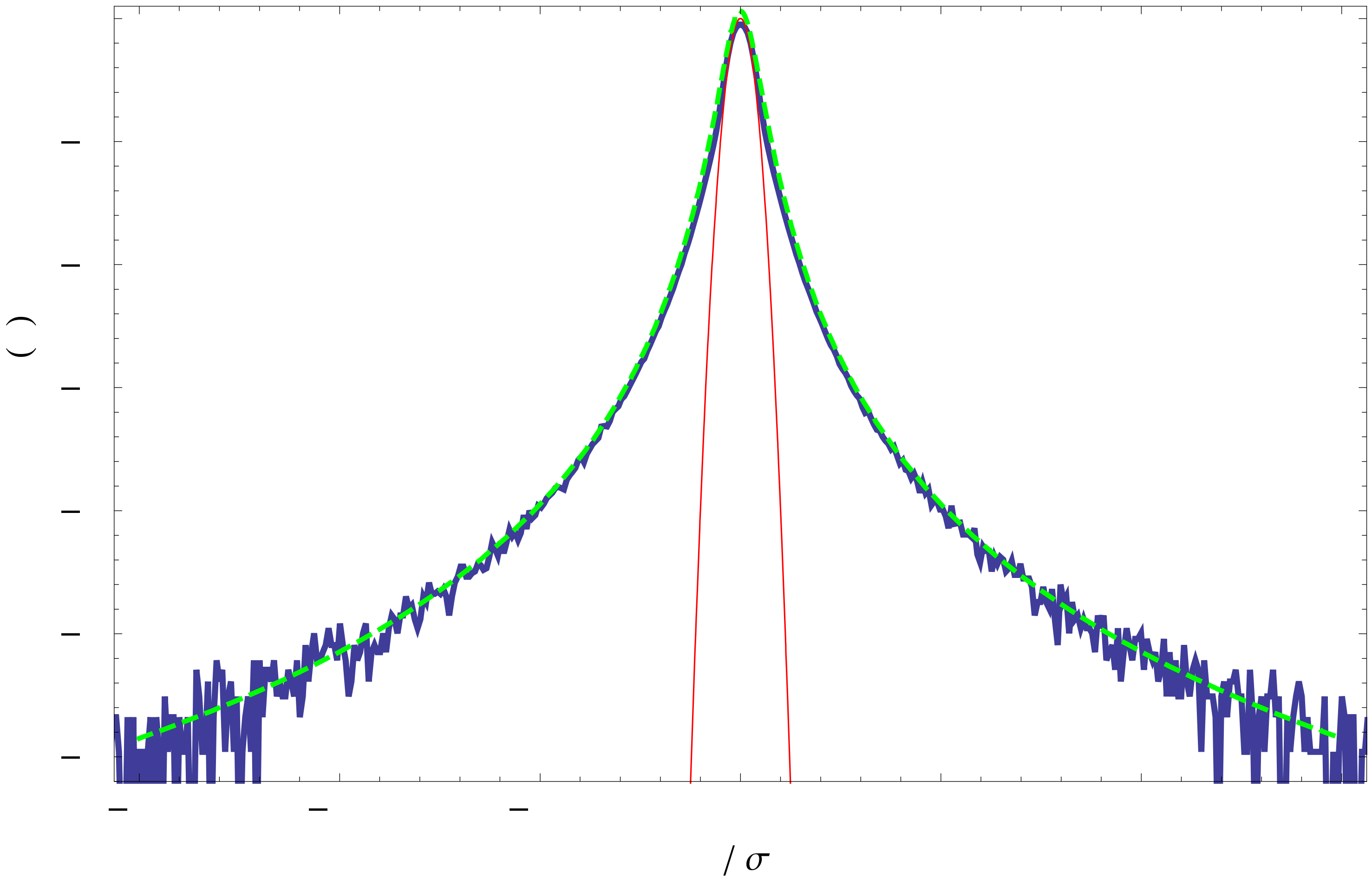
# Figures for acceleration Aq(v)

- Figure (3): Magnitude of the acceleration component |Aq| = Dv |∂v lnPv| generated by the solution of the Schrodinger-Coriolis equation ψv (with Pv = |ψv|2, for Dv = 3, $v_0$ = 1 m/s, n = 3 and lz = 2 (down figure), compared with its form in the absence of a Coriolis force (lz = 0, up figure). The effect of the Coriolis force amounts to enlarge the velocity range where large accelerations are generated.

# Figure for Aq(v) for l=0, and l=2 (n=3)

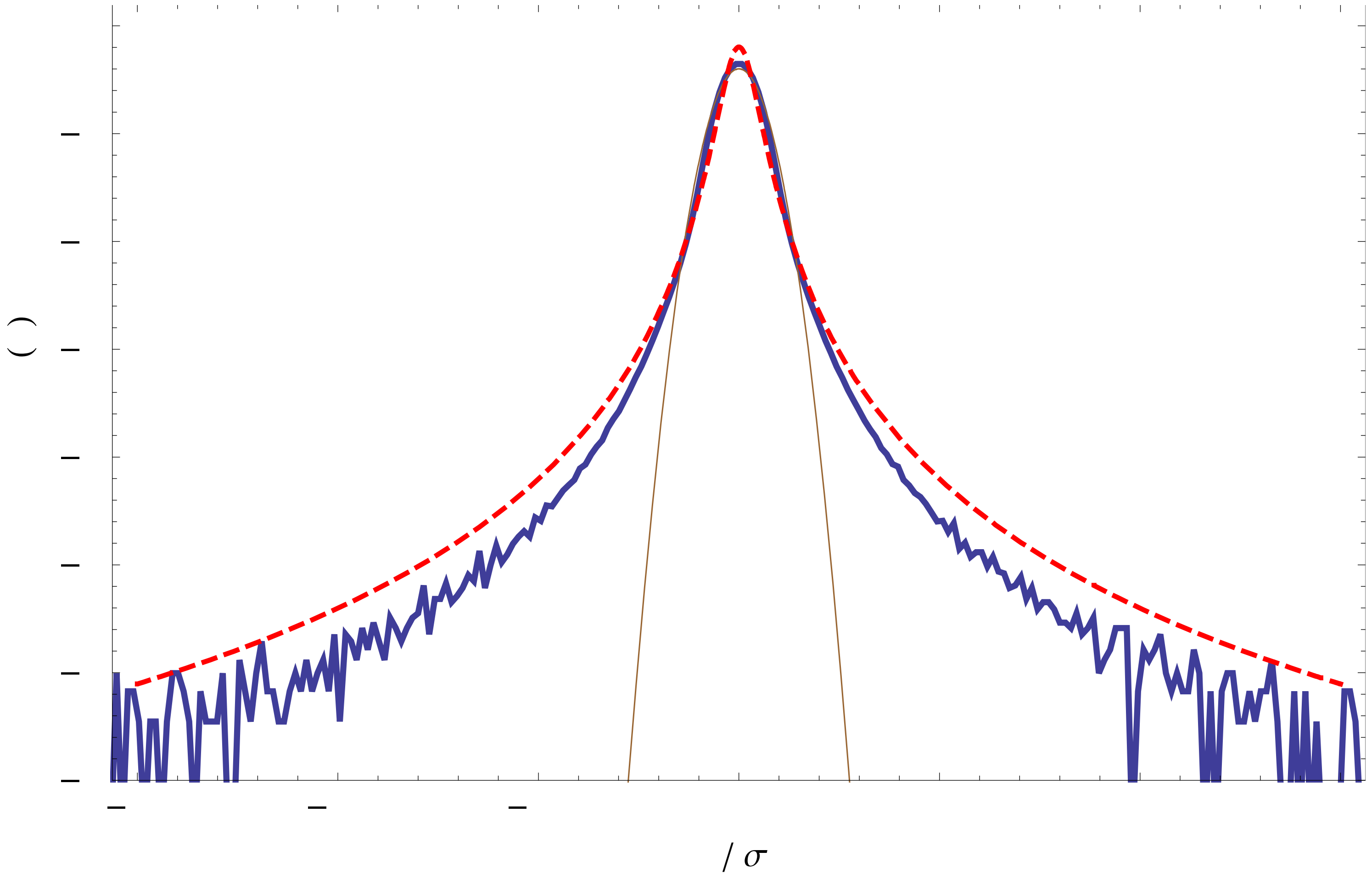


Deformation of P(a) with l=0
brown =Gaussian, green = squared Lorentzian , blue by inversion of HO : recall Isotropic case (no rotation)

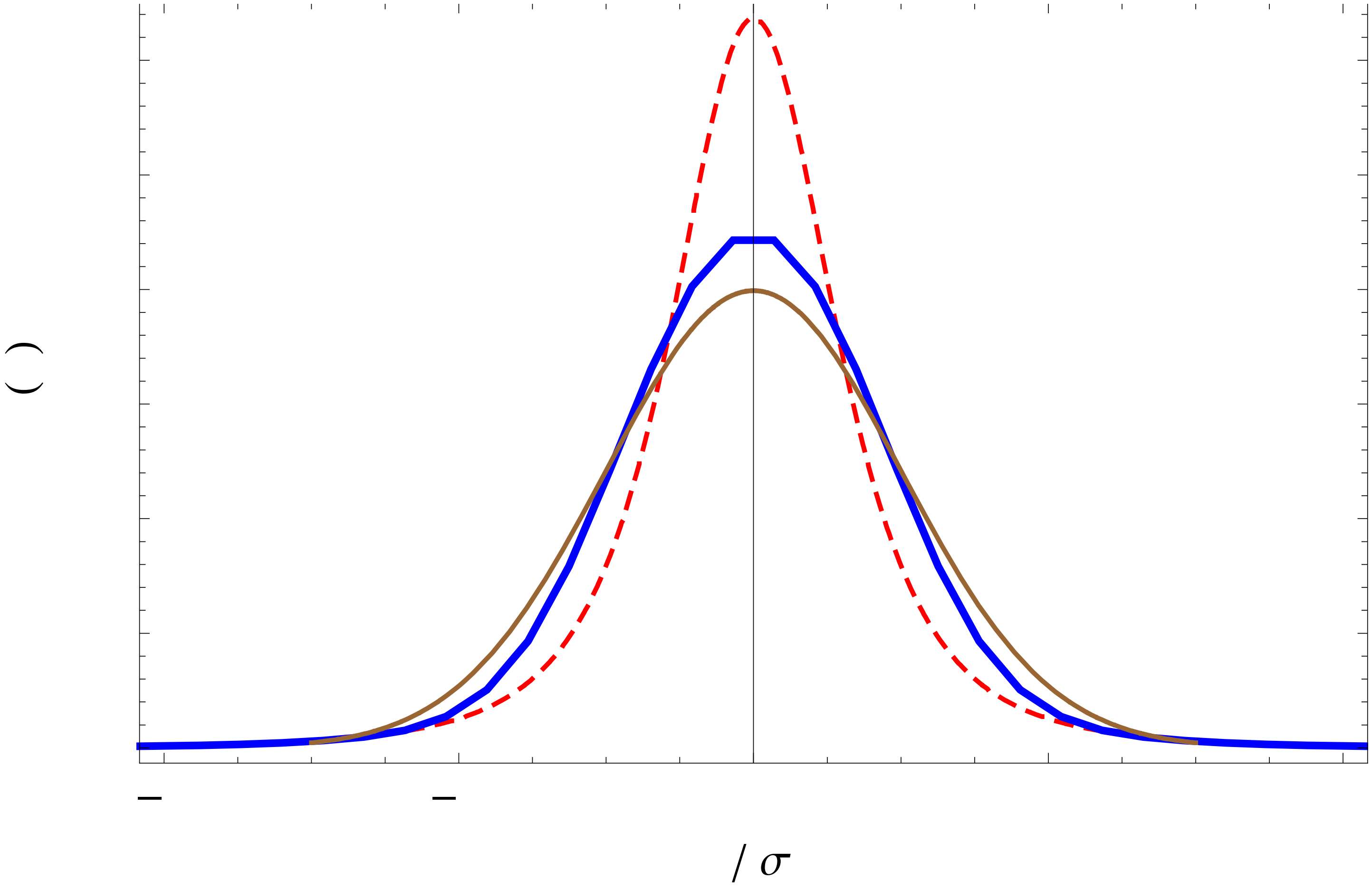


Deformation of P(a) with l=2

brown =Gaussian, red--- = squared Lorentzian , blue by inversion of HO (n=3)



# Deformation of central P(a) with l=2
brown =Gaussian, red--- = squared  Lorentzian , P(a) in blue by inversion of HO (n=3)



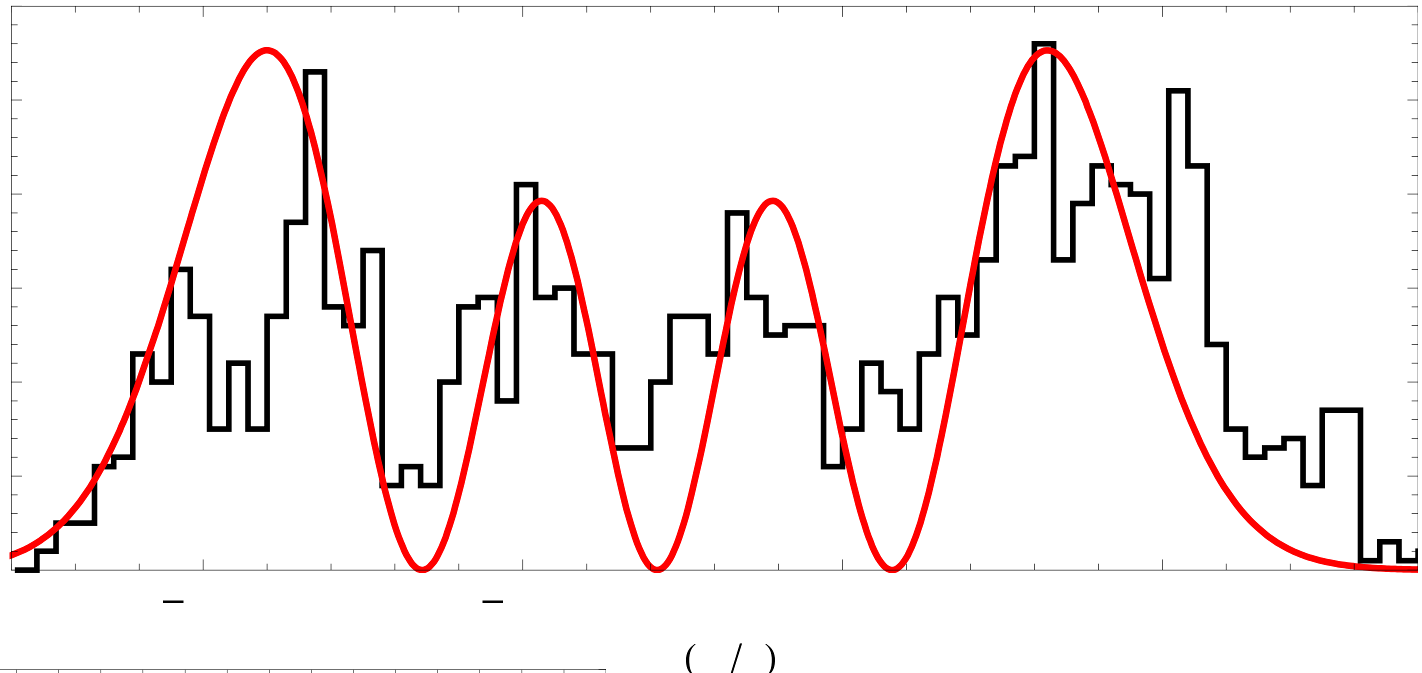
# 3.2 Comparison with data (1)

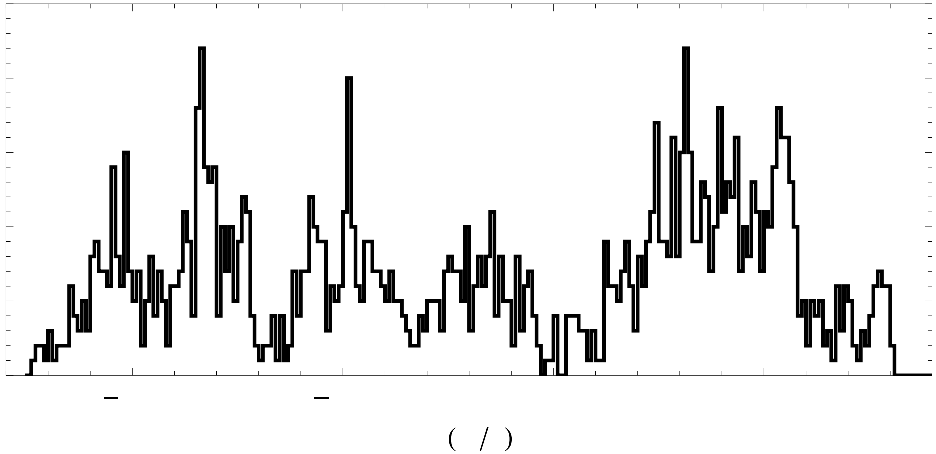
- Various proposals /data :
- Analysis of relevant geophysical data (to be done, but found the right data first )
- Proposal in laboratory experiment : add a rotation in previous experiments. For example with the curve shown on slide 23 we expect if the theory is correct the supression of the two inner secondary peaks in the pdf(vperp).

# 23 bis : Local quantized harmonic v-oscillator Φ(v) shape on a given real data

Seg3398-1-1770: n=3, $v_0$=0.3

All seg ->




Account of Transition cl-qu $|a| < \sigma_a$

# 3.2 Comparison with data here from DNS of A.Pumir et al (2019)

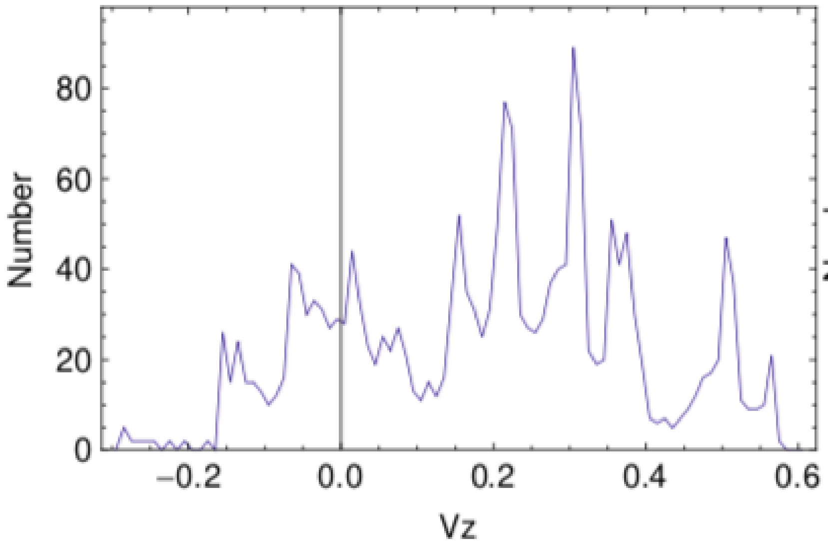
- Conditions of the DNS : (Buria et al) in a cubic domain with $512^3$ grid points +random forcing in t and isotropic in K space , use of GHOST code, but $TL/\tau_\eta$ of order 10 only here. Re=2379 for run HIT , Re= 2645 for run with pure rotation.

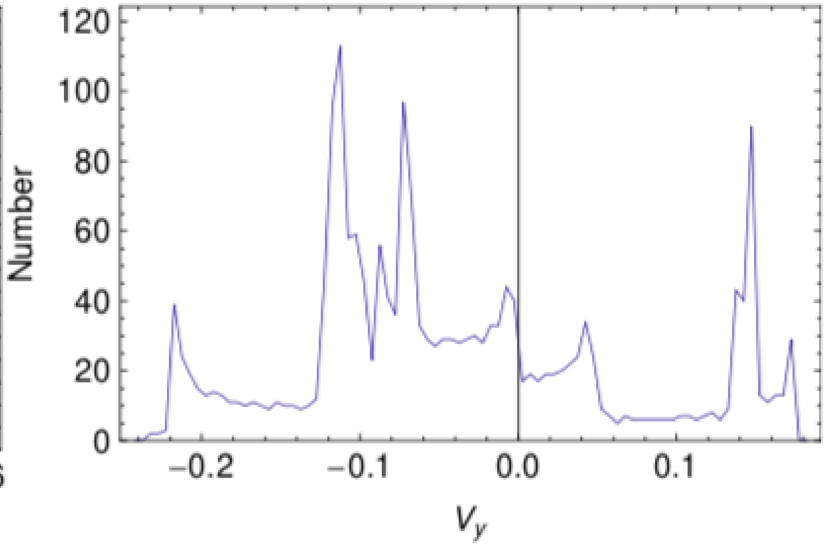
- Comparison of items (but only here on a few Lagrangian trajectories) for run HIT and for run in pure rotation.
- But may be not large enough inertial range of the data …

# Comparison with (DNS) data

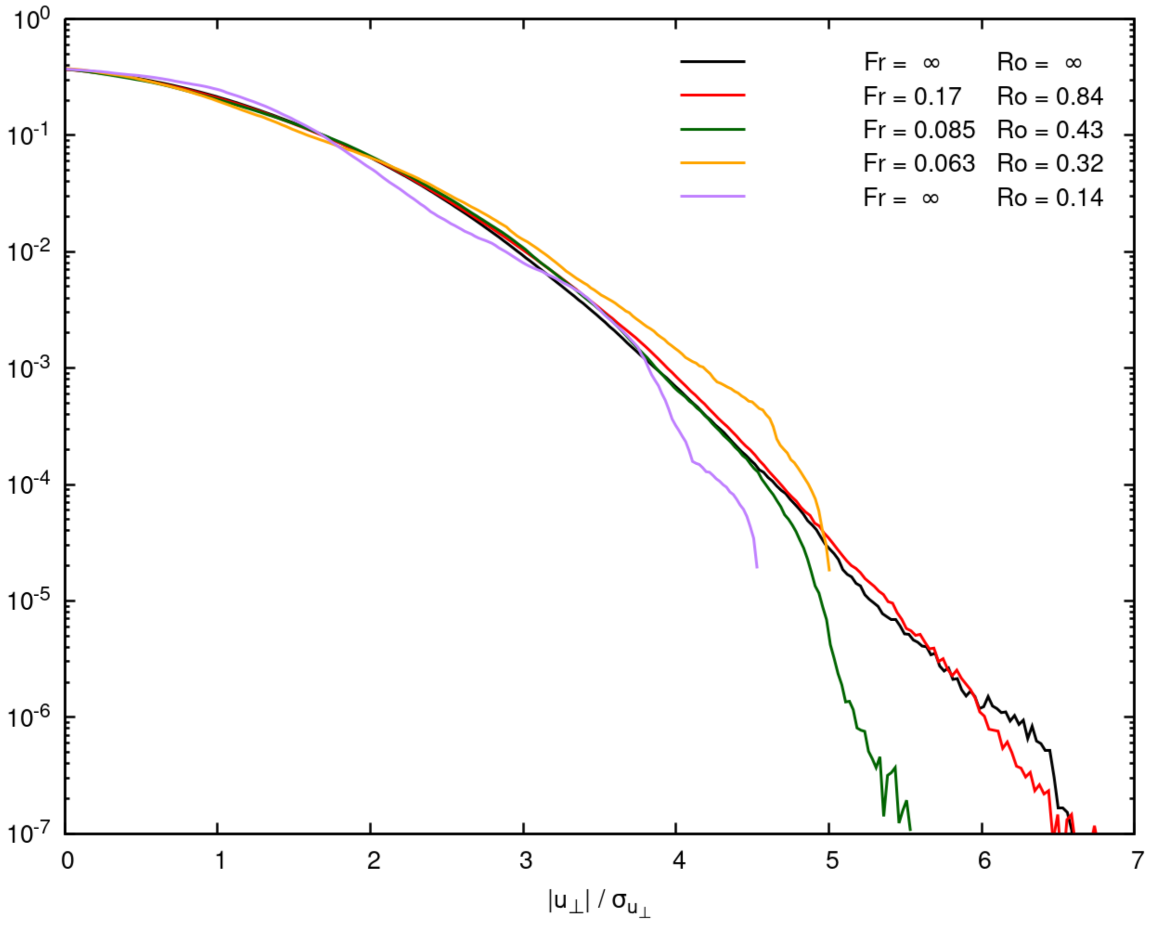
- Figure (5): Examples of typical behavior of velocity PDFs in the no-rotation (left figure) and Coriolis cases (right figure). The no-rotation case shows an alternance of narrow minima and maxima, while the Coriolis case shows, as theoretically expected in the Schrodinger regime, large bands of almost null probability due to the suppression of secondary peaks.
- But to be confirmed on more data (and more turbulent ones)…

# On some L trajectories : Figures for pdf(v)

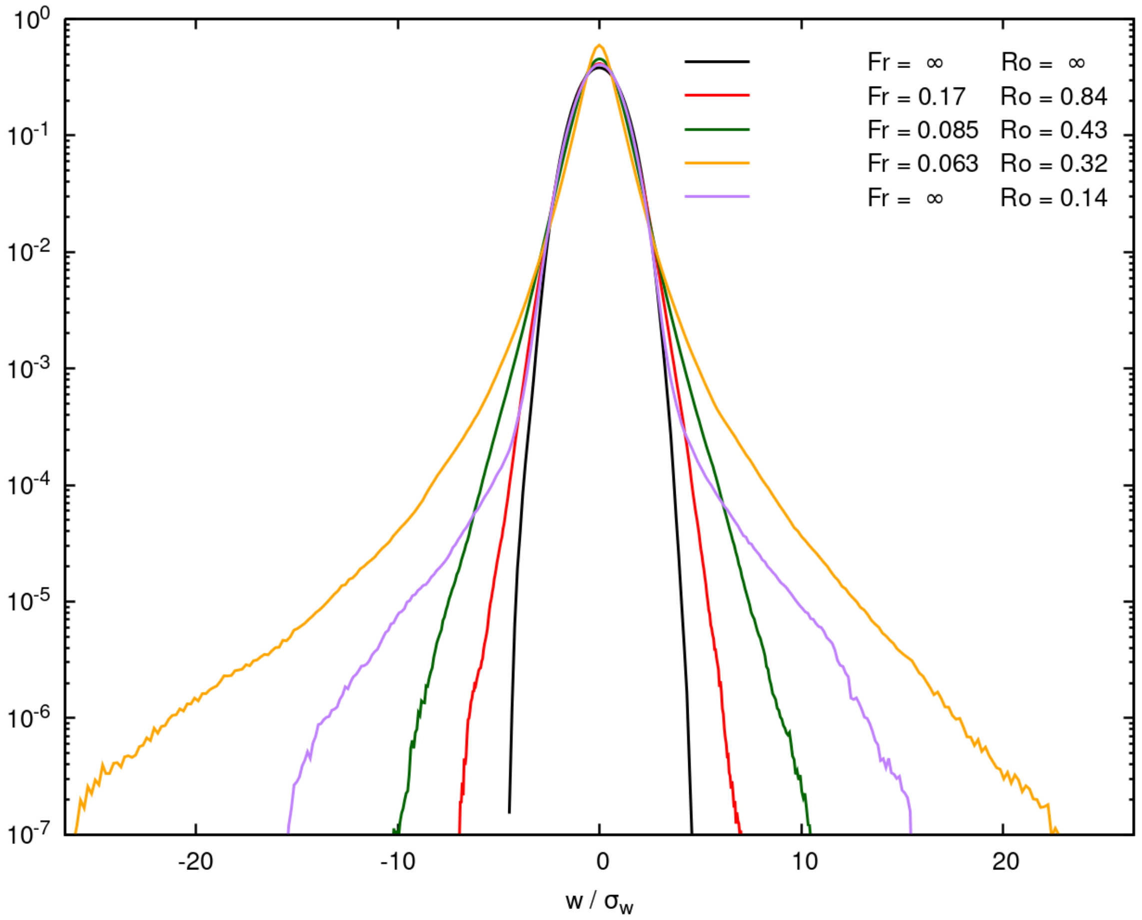


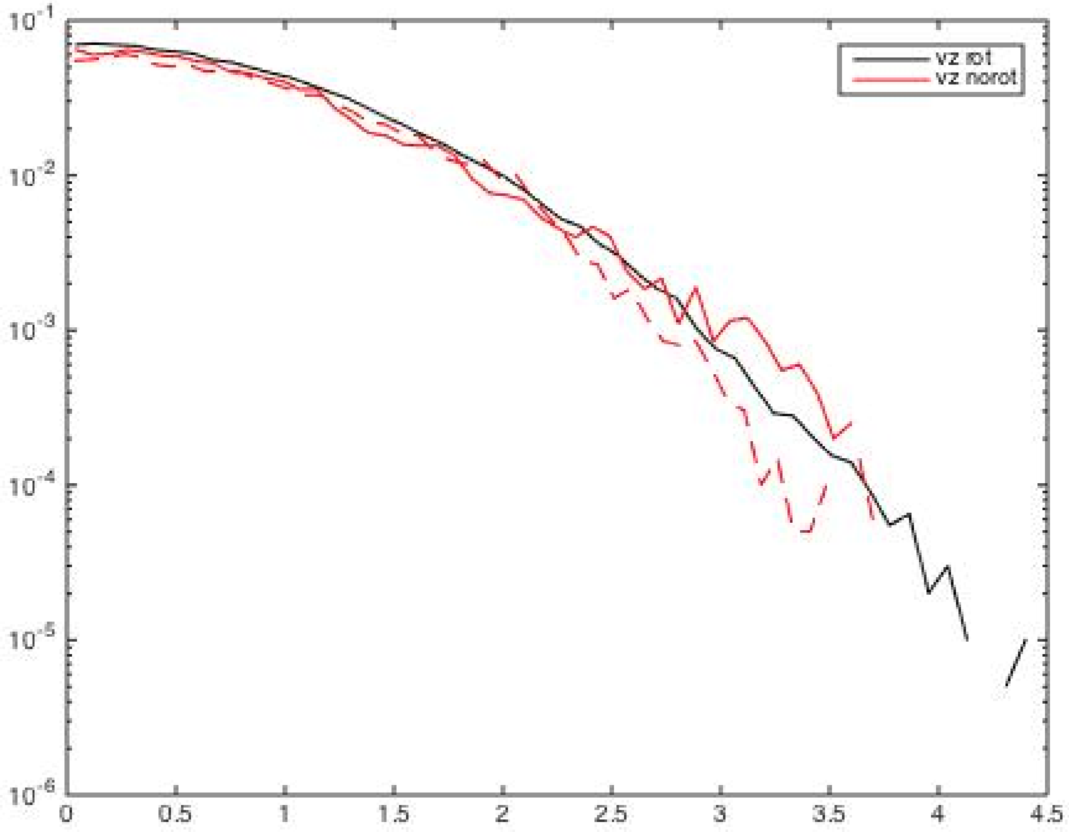
# Fabio Feraccio courtesy , global pdf(vperp) , for different all runs



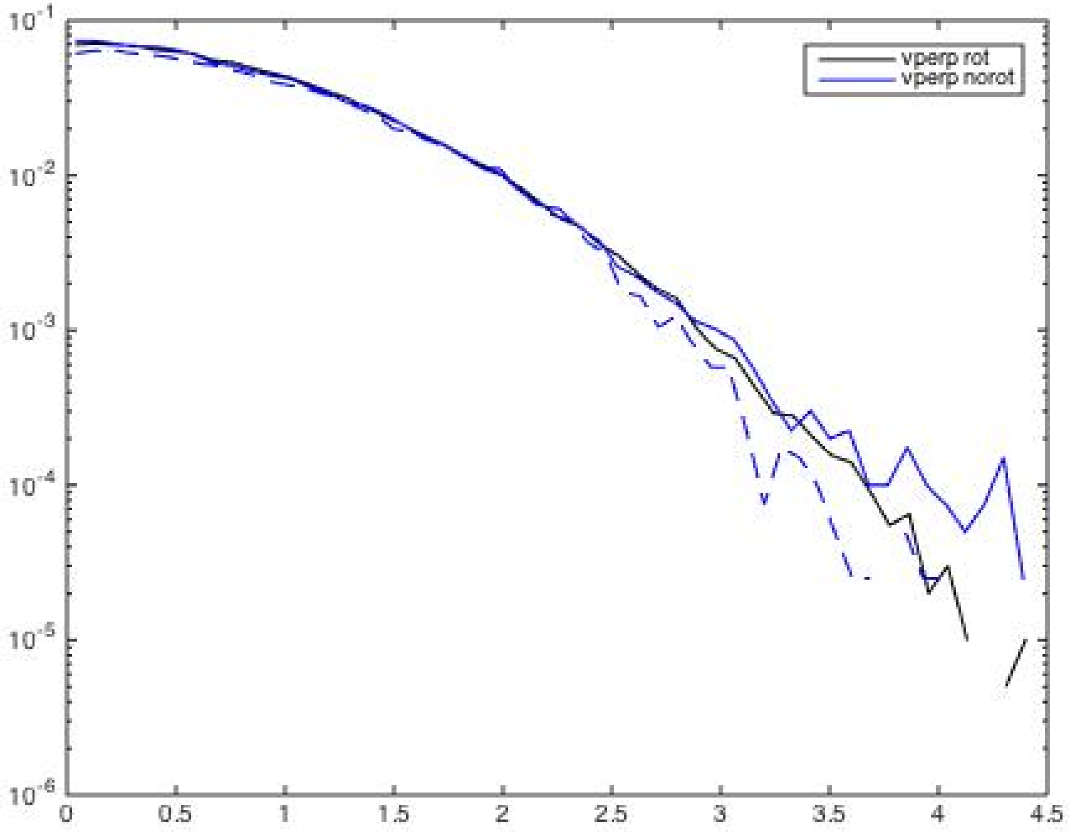
# Fabio Feraccio courtesy , global pdf(vz)



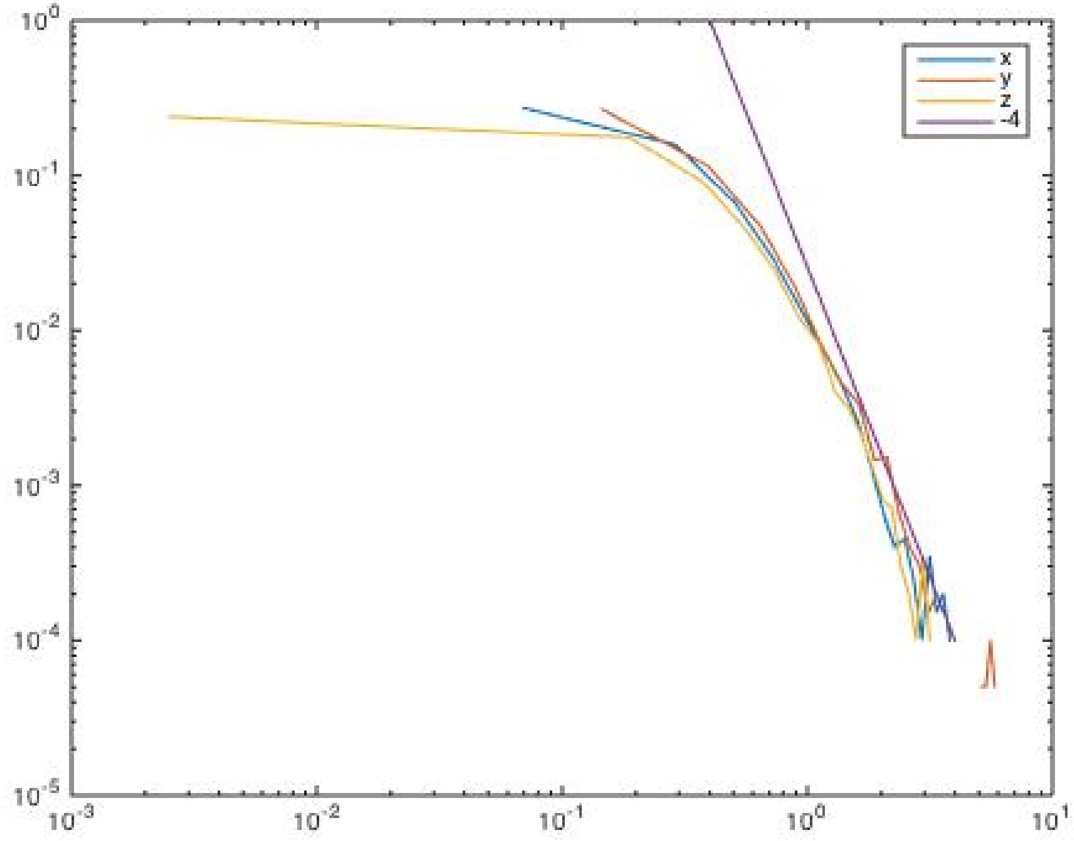
# Our (local) analysis on a few L trajectories for pdf($v_z$) $\Omega$=0 or not



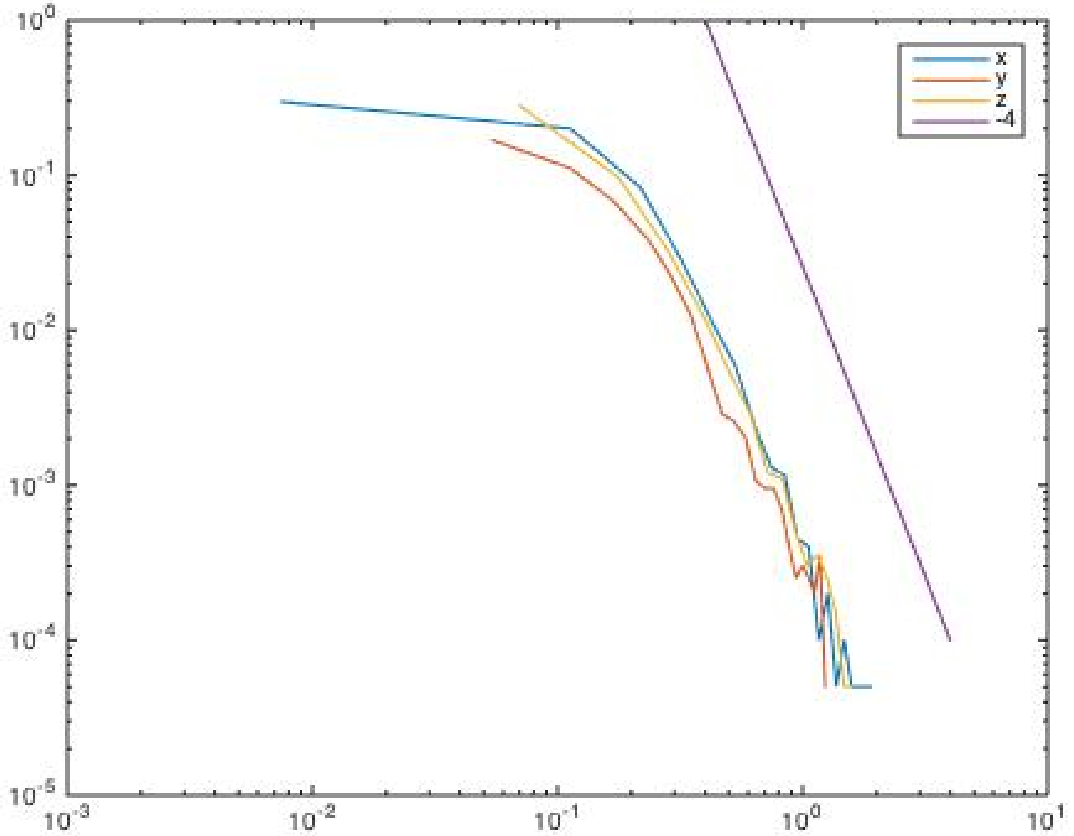
# Idem for pdf($v_{perp}$)



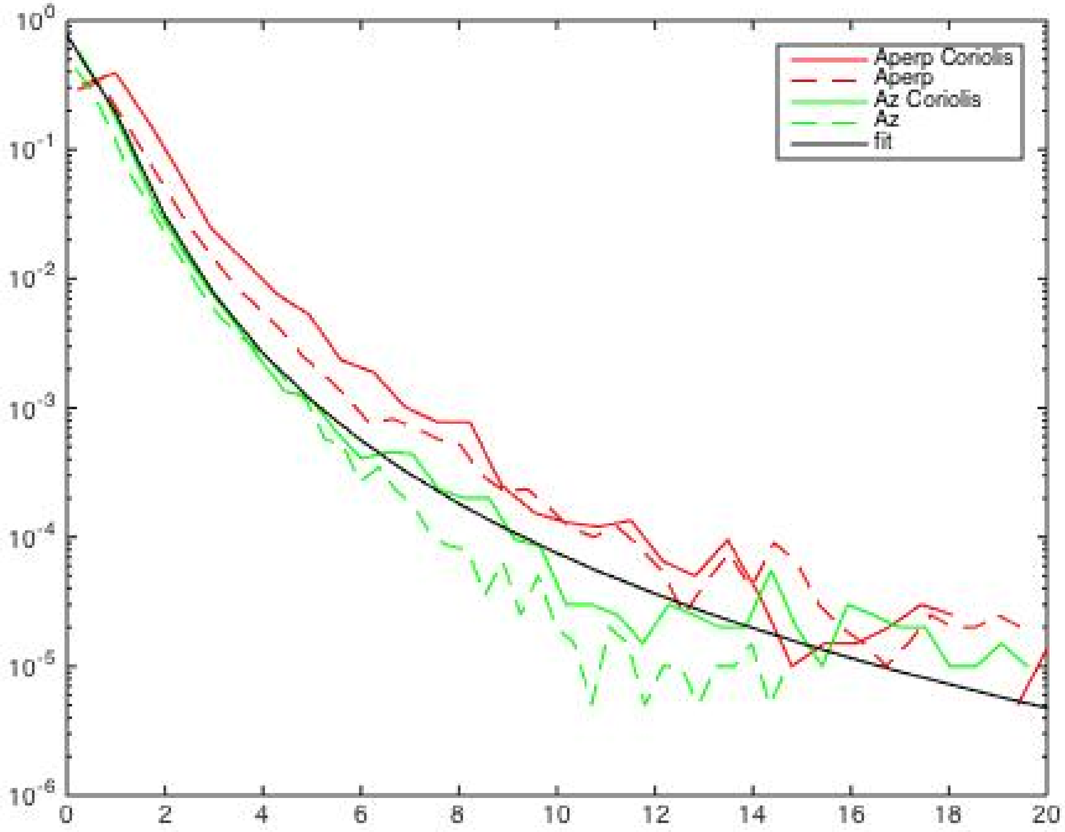
# On L trajectories pdf($a_i$) for $\Omega$=0 (1/ $a_i$ $^4$ law for large $a_i$ i=x,y,z)



# On L trajectories pdf($a_i$) for $\Omega$ non 0 (1/ $a_i$ $^4$ law for large $a_i$ )



# Deformation of P(aperp) in (L) data ?



# Pdf(az) and Pdf(aperp) from Pumir et al paper (aL/aE?) both curves are similar and above the HIT case (---)

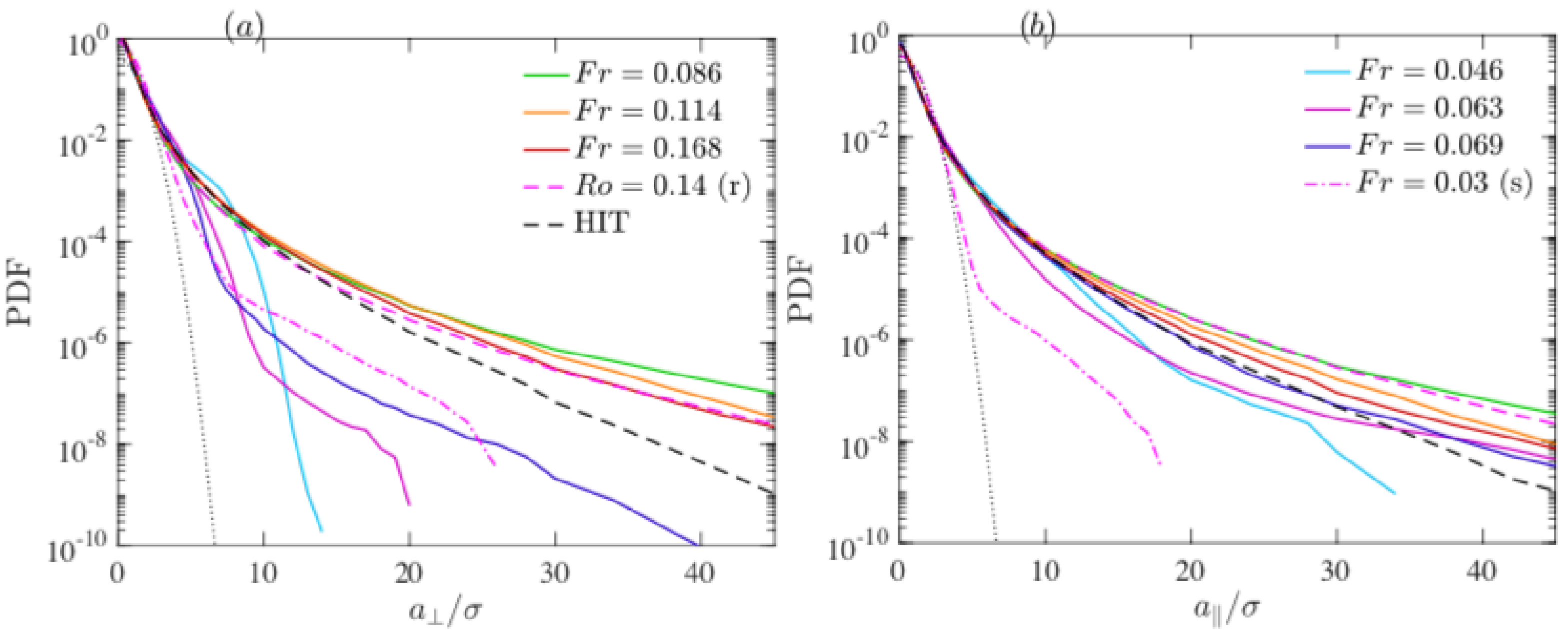



FIG. 3. Standardized probability density functions (PDFs) of (a) the perpendicular (horizontal) and (b) the parallel (vertical) components of acceleration, i.e., $a_\perp$ and $a_\parallel$ respectively. $\sigma$ denotes the corresponding standard deviation. The dotted line shows the standardized Gaussian distribution for comparison. The legend is split over two panels, but applies to each panel individually.

# Conclusions (provisory)

*Predictions done but analysis for the moment done only on very few data: need for a deeper detailed statistical analysis for validation ….and abve all check to be in a suitable turbulent enough regime

*Need of more relevant data from experiments and/or from DNS to test predictions on P(v) , a and P(a)…+anisotropies….

Open pb : new physics involved ? (not included in the DNS) or just a physics already there but hidden, and revealed by the SR approach ?

Next :

- Look also at more involved cases (for theory +predictions)
- Example for Ω not uniform : Ω(x)->Ω (u)? to account for example for a velocity shear….

See later MHD cases on both (v,B) variables