Acceleration scaling and stochastic dynamics of a fluid particle in turbulence

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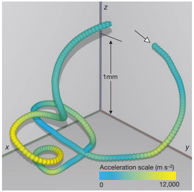
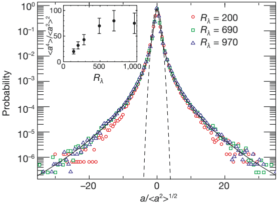
ASTROFLU V

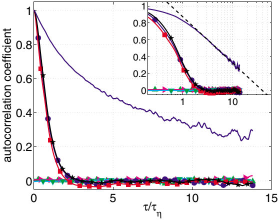
7-8 December 2021



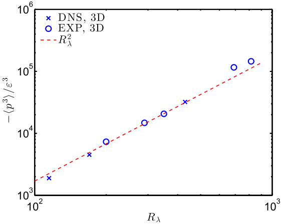
- Non-gaussian PDF
- Scale separation for the correlation
- Asymmetry of the power received/given $\Rightarrow$ irreversibility
- Anomalous scaling of the velocity spectra



La Porta et al. Nature 2001



Mordant et al. PRL 2004

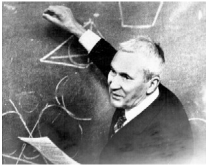


Xu et al. PNAS 2014

Effective dynamics

Navier-Stokes eq.: Collective + dissipative effects

$$a_i = -\partial_i p + \nu \partial^2_{jj} u_i \quad ; \quad p(\mathbf{x}) = \frac{1}{4\pi} \int_V \partial_j u_i \, \partial_i u_j(\boldsymbol{y}) d\boldsymbol{y}/r$$



1st hypothesis of Kolmogorov: Turbulence is universal

$\Rightarrow$ Stochastic model for a fluid-particle dynamics:
effectively account for the interactions with all the other particles

$$da_i = M_i dt + D_{ij} dW_j \quad ; \quad du_i = a_i dt$$

Model based on the conditional statistics of acceleration

- Connection with "intermittency" $\Rightarrow \langle a^2 | \varepsilon \rangle \sim \varepsilon^{3/2} \nu^{-1/2}$
- Stationary dynamics $\Rightarrow \langle a^2 | K \rangle =?$ (Power law ? K^3 or $K^{4.5}$ or $K^{4.6}$?)

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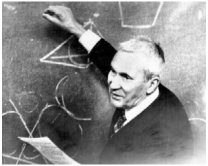

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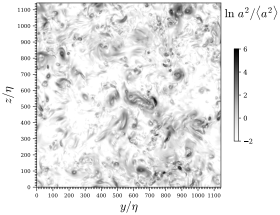
- $\langle a^2 | \varepsilon, K \rangle =?$ never studied!

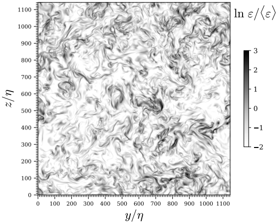
Doubly conditional acceleration statistics

DNS at $Re_\lambda = 250$ ($N = 1024^3$)

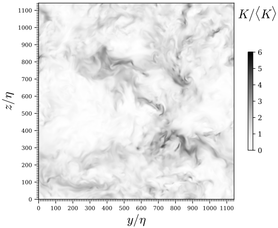
Acceleration



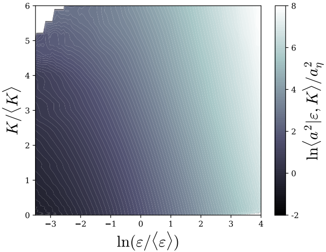
Dissipation



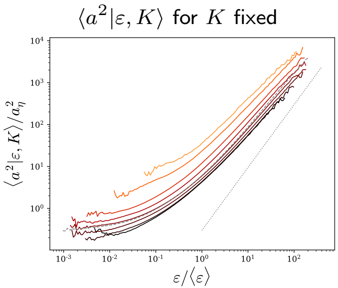
Kinetic energy

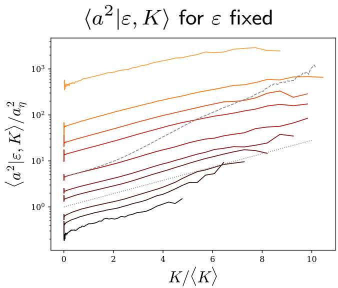


$\langle a^2|\varepsilon, K\rangle \quad \rightarrow$



Doubly conditional acceleration statistics





$$\langle a^2|\varepsilon, K\rangle = A\,\langle a^2|\varepsilon\rangle \exp(\alpha K/\langle K\rangle)$$

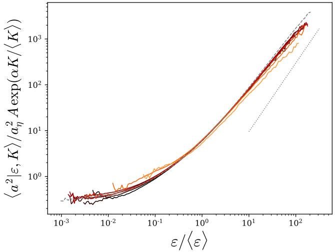
- Exponential dependence with K (not a power law) with growth rate $\alpha = 1/3$

(from independence between fluct. of K and ε and u Gaussian: $A = \left(1 - \frac{2}{3}\alpha\right)^{3/2} \approx 0.69$)

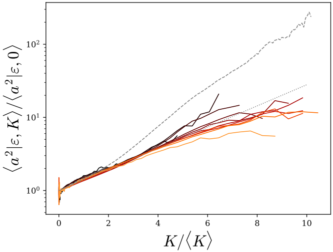
- Scaling law for $\langle a^2|\varepsilon\rangle$?

Doubly conditional acceleration statistics

$\langle a^2|\varepsilon, K\rangle$ for K fixed



$\langle a^2|\varepsilon, K\rangle$ for ε fixed



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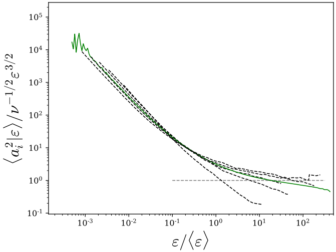


$$\frac{\langle a^2|\varepsilon\rangle}{\varepsilon^{3/2}\nu^{-1/2}} = f(\varepsilon/\langle\varepsilon\rangle, Re_\lambda)$$

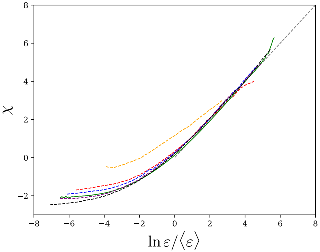
Intermittency $\Rightarrow$ persistence of viscous effects

Barenblatt's Incomplete similarity:
$f(\varepsilon/\langle\varepsilon\rangle, Re_\lambda) = B\left(\varepsilon/\langle\varepsilon\rangle\right)^{\beta}$

$B = B_0 + B_1/\ln(Re_\lambda)$

$\beta = \beta_1/\ln(Re_\lambda)$

$\chi = \frac{1}{3/2+\beta}\ln(1/B\,\langle a^2|\varepsilon\rangle/a_\eta^2)$ vs $\ln\varepsilon/\langle\varepsilon\rangle$



DNS with $Re_\lambda = 40$ to 680

Acceleration conditioned on dissipation

Scaling law for $\langle a^2|\varepsilon\rangle$

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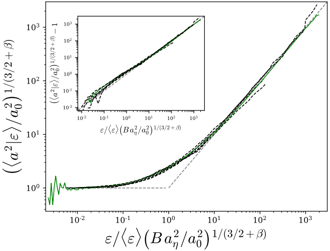
We can go further and obtain a self-similar form introducing $a_0^2/a_\eta^2 = \lim_{\varepsilon\to 0}\langle a^2|\varepsilon\rangle/a_\eta^2 \sim \tau_\eta/\tau_L \sim Re_\lambda^{-1}$



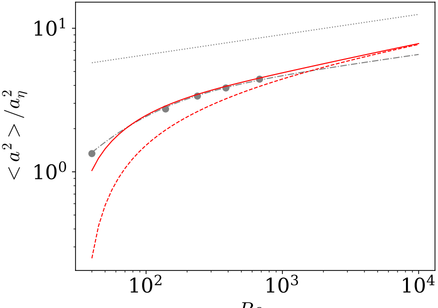
$$\langle a^2|\varepsilon\rangle = Ba_\eta^2\left(\left(\frac{1}{B}\frac{a_0^2}{a_\eta^2}\right)^{1/(3/2+\beta)} + \frac{\varepsilon}{\langle\varepsilon\rangle}\right)^{3/2+\beta}$$

Acceleration variance

$$\langle a^2 \rangle = \int \langle a^2 | \varepsilon \rangle P(\varepsilon) d\varepsilon$$

- $\langle a^2 | \varepsilon \rangle = B a_\eta^2 \left(\left(\frac{1}{B} \frac{a_0^2}{a_\eta^2} \right)^{1/(3/2+\beta)} + \frac{\varepsilon}{\langle \varepsilon \rangle} \right)^{3/2+\beta}$
- $P(\varepsilon)$ Log-normal with parameters $\sigma^2 \approx 3/8 \ln Re_\lambda/10$

Evolution of the acceleration variance vs Re_λ



— $\langle a^2 \rangle = \int \langle a^2 | \varepsilon \rangle P(\varepsilon) d\varepsilon$ x

-- -- -- $Re_\lambda \gg 1$:
$\langle a^2 \rangle \sim a_\eta^2 B (Re_\lambda/10)^{9/64+\beta(3\beta/16+3/8)}$

— — — $Re_\lambda \to \infty$: $\langle a^2 \rangle \sim a_\eta^2 Re_\lambda^{9/64}$

● DNS data Yeung et al 2006

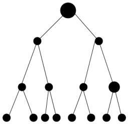
— . — empirical formula Sawford et al. 2003:
$\langle a^2 \rangle \sim a_\eta^2 1.9 \frac{Re_\lambda^{0.135}}{1 + 85/Re_\lambda^{1.135}}$

Acceleration as a multiplicative process

Finally we obtain:

$$\langle a^2|\varepsilon, K\rangle/a_\eta^2 = C\exp(\alpha K/\langle K\rangle)\left(\frac{\varepsilon}{\langle\varepsilon\rangle}\right)^\gamma \quad C = AB\ ;\ \gamma = 3/2+\beta$$

What is the physical interpretation ?



- Locally-space-averaged dissipation as a multiplicative process:

$$\varepsilon_\ell = \langle\varepsilon\rangle\prod_{i=1}^{n}\xi_i \quad ; \quad \ell = L\lambda^n$$

- Similarly, multiplicative process for coarse-grained acceleration:

$$a_\ell^2 = a_0^2\prod_{i=1}^{n}\theta_i \qquad \text{with} \qquad \theta_i = \exp\left(\frac{\alpha}{\langle K\rangle}\frac{1}{2}u_i^2 + \gamma\ln\xi_i\right) \Rightarrow$$

$$a_\ell^2 = a_0^2\exp\left(\frac{\alpha}{\langle K\rangle}\sum_{i=1}^{n}\frac{1}{2}u_i^2 + \gamma\sum_{i=1}^{n}\ln\xi_i\right)$$

$\Rightarrow$ Scale similarity with sweeping effects from eddies of size ℓ_i to balance the intense local acceleration induced by ε_ℓ

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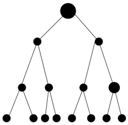

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with $n = \ln(\eta/L)/\ln(\lambda) \sim \ln Re_\lambda$

$$a_\ell^2 = a_0^2\exp\left(\frac{\alpha}{\langle K\rangle}\underbrace{\sum_{i=1}^{n}\frac{1}{2}u_i^2}_{K} + \gamma\underbrace{\sum_{i=1}^{n}\ln\xi_i}_{\ln\varepsilon/\langle\varepsilon\rangle}\right)$$

$\Rightarrow$ Scale similarity with sweeping effects from eddies of size ℓ_i to balance the intense local acceleration induced by ε_ℓ

Stochastic dynamics of a fluid particle

We propose a stochastic model for the tracer using 4 assumptions:

- H1: Assume the dynamics can be expressed as a closed model (universality)

$$da_i = M_i dt + D_{ij} dW_j \quad ; \quad du_i = a_i dt$$

Ito Formula $\Rightarrow$

$$da^2 = (2a_i M_i + D_{ij} D_{ij})\, dt + 2a_i D_{ij} dW_j$$

- H2: The instantaneous acceleration is given by the doubly conditional variance (the remaining degree of freedom can be discarded)

$$a^2 = a_\eta^2 C \left(\frac{\varepsilon}{\langle \varepsilon \rangle} \right)^\gamma \exp\left(\alpha \frac{K}{\langle K \rangle} \right)$$

Taylor expansion: $da^2 = a^2 \left(\alpha \frac{dK}{\langle K \rangle} + \gamma \frac{d\varepsilon}{\varepsilon} + \frac{\gamma(\gamma-1)}{2} \frac{d\varepsilon^2}{\varepsilon^2} \right)$

With $dK = a_i u_i dt = P dt$ and $d\varepsilon = \varepsilon \Pi dt + \varepsilon \Sigma dW$

$$da^2 = a^2 \left[\frac{\alpha}{\langle K \rangle} P + \gamma \Pi + \frac{\gamma(\gamma-1)}{2} \Sigma^2 \right] dt + \gamma a^2 \Sigma dW$$

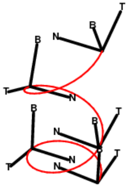
$\Rightarrow$ Identification between the two equations

- H3: Introduction of a non-diagonal diffusion tensor:

$$D_{ij} = c_1 \delta_{ij} + c_2 \epsilon_{ijk} \omega_k \quad ; \quad \omega_k = \epsilon_{ijk} u_i a_j$$

with the "maximum winding hypothesis":

$$D_{ij} = \sqrt{\frac{\gamma^2}{4} \Sigma^2} \left[\sqrt{a_T^2} \delta_{ij} + \sqrt{a_N^2} \epsilon_{ijk} b_k \right]$$



- H4: Dissipation rate along the trajectory is given by the non-Markovian log-normal process proposed by L. Chevilard
Logarithmic correlation of $\varepsilon \Rightarrow$ cascade picture

$$d\varepsilon = \varepsilon \underbrace{\left(-\ln \frac{\varepsilon}{\langle \varepsilon \rangle} + \frac{\sigma^2}{2\Lambda^2} \left(\frac{\tau_\varepsilon}{\tau_\eta} - \Lambda^2 \right) + \frac{\sigma}{\Lambda} \Gamma \tau_\varepsilon \right)}_{\Pi} dt/\tau_\varepsilon + \varepsilon \underbrace{\sqrt{\frac{\sigma^2}{\Lambda^2 \tau_\eta}}}_{\Sigma} dW$$

with σ^2 the variance of $\ln \varepsilon$: $\sigma^2 \approx 3/8 \ln Re_\lambda / 10$

and Γ the non-Markovian term: $\Gamma = -\frac{1}{2} \int_{-\infty}^{t} (t - s + \tau_\eta)^{-3/2} dW(s)$

(Λ a normalization constant)

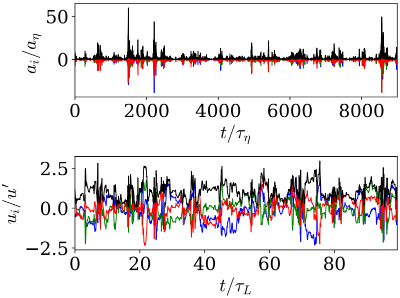
Stochastic dynamics of a fluid particle

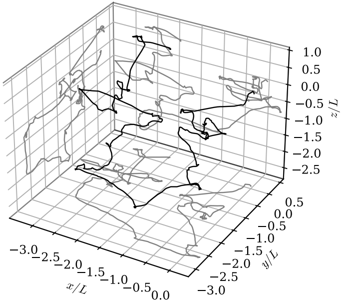
Finally we obtain our stochastic equation (no free parameters: they are all determined from DNS!)

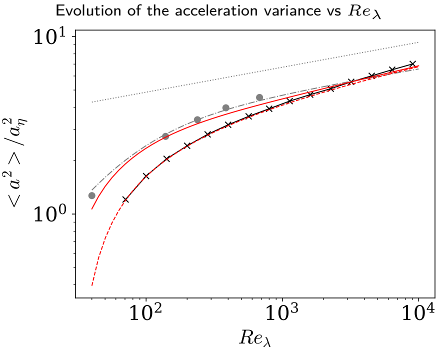
$$da_i = \left[\frac{\alpha}{2\langle K\rangle}\left(a_i\left(5.2P + \frac{K}{\tau_\varepsilon}\right) - 4.2a^2u_i\right) - a_i\left(\ln\left(\frac{a^2}{a_\eta^2}\right) + \hat{\Gamma}_*\right)\frac{1}{2\tau_\varepsilon} - \frac{\sigma_*^2}{\tau_\eta}\frac{a_T^2}{a^2}a_i\right]dt$$

$$+\sqrt{\frac{\sigma_*^2}{\tau_\eta}}\left[\sqrt{a_T^2}\delta_{ij} + \sqrt{a_N^2}\epsilon_{ijk}b_k\right]dW_j \qquad ; \qquad du_i = a_i dt$$

One realization at $Re_\lambda = 1000$







× Stochastic model —— $\langle a^2 \rangle = \int \langle a^2 | \varepsilon \rangle P(\varepsilon) d\varepsilon$ ×

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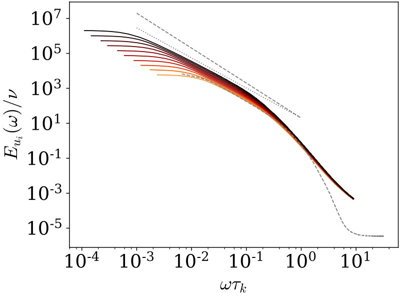
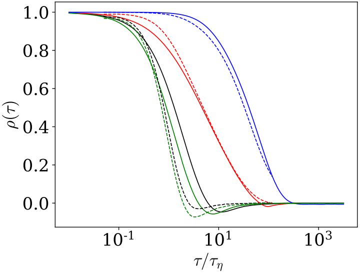
Stochastic dynamics of a fluid particle

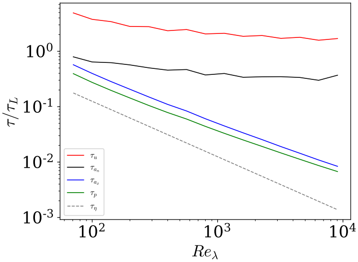
model $Re_\lambda = 400 \to 9000$
$+$ comparison from DNS $Re_\lambda = 400$

Lagrangian velocity spectra
Deviation from Hinze spectra, anomalous scaling



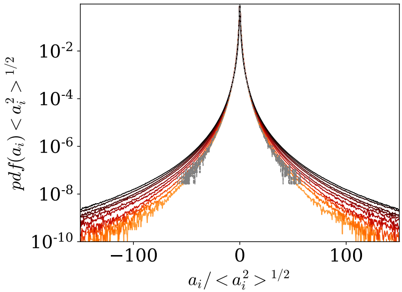
Scale separation between the correlation for norm and the components of the acceleration



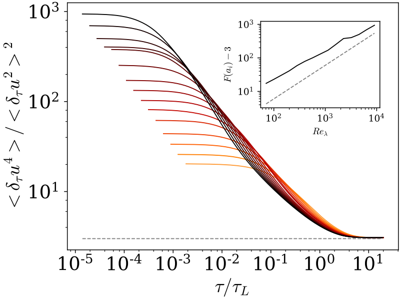


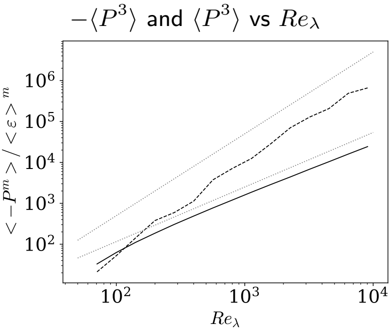
Model for $Re_\lambda = 400 \to 9000$
$+$ comparison from DNS $Re_\lambda = 400$

Non gaussian PDF



Increase of the velocity increments flatness with the scale and the Reynolds number





$$\langle P^2\rangle/\langle\varepsilon\rangle^2 \sim Re_\lambda^{4/3} \text{ and } -\langle P^3\rangle/\langle\varepsilon\rangle^3 \sim Re_\lambda^2$$

Skewness of the mechanical power
$\Rightarrow$ time irreversibility of the dynamics

(connected to non-markovianity + non-diagonal diffusion)

Summary / Conclusion

DNS of Navier-Stokes:

- Doubly conditional variance: $\langle a^2|\varepsilon, K\rangle / a_\eta^2 = C \exp(\alpha K/\langle K\rangle) \left(\frac{\varepsilon}{\langle\varepsilon\rangle}\right)^\gamma$
- Multiplicative process for the acceleration accounting for the sweeping effects
- $\alpha = 1/3$ and for 2D ? or non isotropic turbulence ?
- Relation between force/power/energy

Stochastic dynamics of fluid particles:

- Only 4 reasonable hypothesis
- Non-gausssianity, long-range correlations, anomalous scaling and time irreversibility.
- Good agreement with the DNS (no free parameters).
- What next ?
  - Extension to non-stationnary / non-homogenous turbulence ?
  - Improve the high frequency/dissipative part
  - Theoretical/mathematical analysis of the model
  - Application to LES / RANS modeling and "multiphysics" coupling

You can check the preprint: https://hal.archives-ouvertes.fr/hal-03408311

Calculation of the history integral

$$\Gamma(t) = \int_{-\infty}^{t} (t - s + \tau_c)^{-3/2} dW(s)$$

$$\Gamma_n = \sum_{m=0}^{N_{hist}} (s_m + \tau_c)^{-3/2}\, dW_{n-m}$$

$$\approx \sum_{j=1}^{N} (\overline{s}_j + \tau_c)^{-3/2}\, \overline{dW}_j$$