Testing wave turbulence theory for Gross-Pitaevskii system

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Gross-Pitaevskii equation (GPE)

Nonlinear Schrödinger equation (NLSE)

$$i\frac{\partial\psi(\boldsymbol{r},t)}{\partial t} + \nabla^2\psi(\boldsymbol{r},t) + s\left|\psi(\boldsymbol{r},t)\right|^2\psi(\boldsymbol{r},t) = 0$$

- $\psi(\boldsymbol{r},t)$, complex scalar function
- $\boldsymbol{r} \in \mathcal{R}^d$, $d \leq 3$, time $t \in \mathcal{R}$.
- $s = 1$ and $s = -1$ correspond to the focusing or defocusing GPE respectively.

Physical systems described by Gross-Pitaevskii equation

- Bose-Einstein Condensates (BECs) [Pitaevskii-book]
- nonlinear optical systems [Newell :1992il, DNPZ-bec]
- cosmological evolution of early Universe [Zurek1996].
- superfluid flows at almost zero temperatures [Barenghi01-proj, koplik]

GPE for Bose-Einstein condensates

$$i\frac{\partial\psi(\boldsymbol{r},t)}{\partial t}+\nabla^2\psi(\boldsymbol{r},t)-|\psi(\boldsymbol{r},t)|^2\,\psi(\boldsymbol{r},t)=0$$

- 3D $\psi(\boldsymbol{r},t)$ wave function, L-periodic cube $\mathbb{T}_L$
- nonlinear wave system : random mutually interacting waves with a broadband spectrum
- additional terms : forcing, damping, potential
- conservation laws :
  - number of particles $N=\frac{1}{V}\int|\psi(\mathbf{r},t)|^2d\mathbf{r}$
  - energy $H=\frac{1}{V}\int\left[|\nabla\psi(\mathbf{r},t)|^2+\frac{1}{2}|\psi(\mathbf{r},t)|^4\right]d\mathbf{r}$
- condensate fraction : $C_0=\frac{|\langle\psi(\boldsymbol{r},t)\rangle|^2}{N}$, $\psi(\boldsymbol{r},t)=\langle\psi(\boldsymbol{r},t)\rangle+\psi'(\boldsymbol{r},t)$
- healing length : $\xi=\frac{1}{\sqrt{N}}$

Weak wave turbulence theory (WWTT)

- statistical spectral theory

$$\hat{\psi}_{\mathbf{k}} = \hat{\psi}(\mathbf{k}, t) = \frac{1}{V} \int_{\mathbb{T}_L} \psi(\boldsymbol{r}, t) e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}, \quad \psi(\boldsymbol{r}, t) = \sum_{\boldsymbol{k}} \hat{\psi}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}$$

- wave-action spectra : $n_{\boldsymbol{k}}(t) = n(\mathbf{k}, t) = \left(\frac{L}{2\pi}\right)^3 \langle |\hat{\psi}_{\mathbf{k}}|^2 \rangle$
- weak non-linearity ; absent of condensation ;
  four-wave system
- set time scale T to be intermediate between linear and non-linear scales
- use the random phase and amplitude (RPA) assuption (non-Gaussianity)
- take large box limit $L \to \infty$, than the limit $T \to \infty$

Weak wave turbulence theory (WWTT)

wave-kinetic equation (WKE)

$$\frac{d}{dt}n_{\mathbf{k}} = 4\pi \int \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\omega_{\mathbf{k}}+\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}-\omega_{\mathbf{k}_3}) \Big[n_{\mathbf{k}_1}n_{\mathbf{k}_2}n_{\mathbf{k}_3}+n_{\mathbf{k}}(n_{\mathbf{k}_2}n_{\mathbf{k}_3}-n_{\mathbf{k}_1}n_{\mathbf{k}_3}-n_{\mathbf{k}_1}n_{\mathbf{k}_2})\Big]d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3,$$

- assume that $n_{\boldsymbol{k}}$ is isotropic in $\boldsymbol{k}$ -space
- pass to the frequency variable : $n_{\mathbf{k}}(t) = n_\omega(t) = n(\omega, t)$, $\omega_{\mathbf{k}} = |\boldsymbol{k}|^2$

Weak wave turbulence theory (WWTT)

wave-kinetic equation (WKE) [Semikoz and Tkachev, 1995(PRL)]

$$\frac{d}{dt}n_\omega = \frac{4\pi^3}{\sqrt{\omega}} \int S(\omega,\omega_1,\omega_2,\omega_3)\delta^{23}_{1\omega}\, n_\omega n_1 n_2 n_3 \left(n_\omega^{-1} + n_1^{-1} - n_2^{-1} - n_3^{-1}\right) d\omega_1 d\omega_2 d\omega_3$$

$S(\omega,\omega_1,\omega_2,\omega_3) = \min\left(\sqrt{\omega}, \sqrt{\omega_1}, \sqrt{\omega_2}, \sqrt{\omega_3}\right)$, $\omega = |\boldsymbol{k}|^2$
$\delta^{23}_{1\omega} = \delta(\omega + \omega_1 - \omega_2 - \omega_3)$

- conservation law : $N = 2\pi \int_0^\infty \omega^{1/2}\, n_\omega\, d\omega$, $H = 2\pi \int_0^\infty \omega^{3/2}\, n_\omega\, d\omega$
- ω-space continuous, $L \to \infty$
- thermodynamic equilibrium solutions : $n_\omega = \dfrac{T}{\omega + \mu}$
- non-equilibrium stationary power-law, Kolmogorov-Zakharov (KZ) spectra $n_\omega = C\omega^{-x}$
  direct cascade of energy : $x = 3/2$; inverse cascade of particles : $x = 7/6$

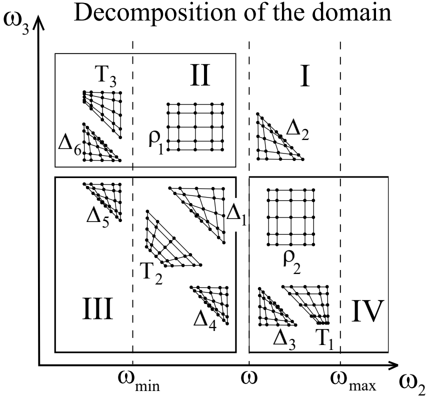
Motivation : testing wave turbulence theory

- temporal evolution [Zhu, Semisalov, Krstulovic, Nazarenko, 2021(arXiv :2111.14560)]
  - rigorous mathematical justification, $\delta \cdot T_{kin}$, $\delta \ll 1$ [Deng and Hani, 2021(arXiv :2106.0981)]
  - wave-action spectrum, probability density function (PDF)
- stationary solutions
  - direct cascade : experiments by [Navon et al., 2016(Nature)], mystery -1.5 law
  - inverse cascade : verify self-similar solution obtained by WKE [Semisalov et al., 2021(Communications in Nonlinear Science and Numerical Simulation)]
  - inverse cascade : achieve stationary solution, predict wave-action spectrum with flux
- self-similar solutions
  - second kind of self-similar solutions predicted by WWTT

Numerical method for the WKE

- collocation method, rational barycentric interpolations
  accounts for singularities, allows one to obtain highly-accurate results
- integration adapts to smoothness, exponential convergence [Semisalov et al., 2021]

Decomposition of the domain



I-IV include different values of the kernel

- nonuniform grid points using special mappings of Chebyshev nodes

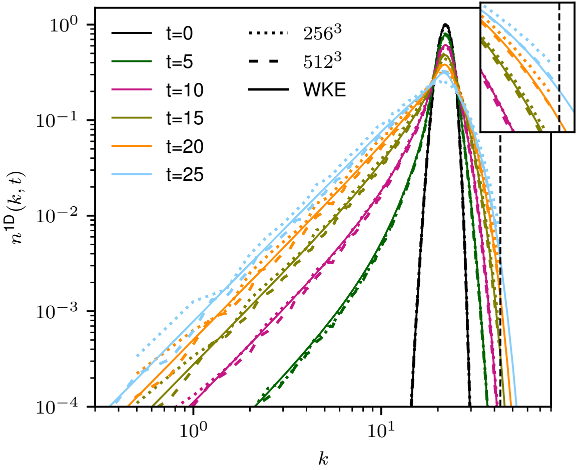
Numerical setup

- Initial 1D wave-action spectrum for GPE and WKE

$$n^{1D}(k,0) = g_0 \exp\left(\frac{-(k-k_s)^2}{\sigma^2}\right)$$

  with $g_0 = 1$, $k_s = 22$, $\sigma = 2.5$
  GPE : random initial phases
- Numerical methods of GPE :
  - integrated with a pseudo-spectral numerical method on cubic, triply-periodic domain
  - Exponential time differencing version of the Runge–Kutta scheme of fourth order (ETD4RK) is employed for time evolution

Evolution of 1D wave-action Spectrum for short time



Time evolution of $n^{1D}(k,t)$ for short time. $T_{kni} = 15$

Evolution of 1D wave-action Spectrum for short time

centroids of $n^{1D}(k,t)$ and $E^{1D}(k,t)$ and their typical widths

$$K_N(t) = \frac{1}{N_c(t)} \int_0^{k_{\text{cutoff}}} k n^{1D}(k,t) \mathrm{d}k,$$

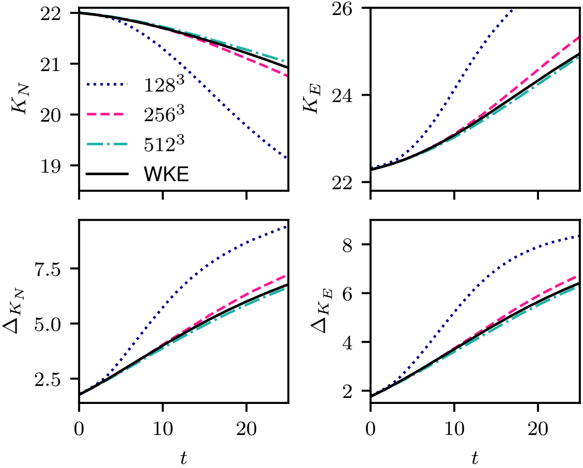
$$\Delta_{K_N}(t) = \sqrt{\frac{1}{N_c(t)} \int_0^{k_{\text{cutoff}}} (k - K_N)^2 n^{1D}(k,t) \mathrm{d}k},$$

$$K_E(t) = \frac{1}{H_c(t)} \int_0^{k_{\text{cutoff}}} k E^{1D}(k,t) \mathrm{d}k,$$

$$\Delta_{K_E}(t) = \sqrt{\frac{1}{H_c(t)} \int_0^{k_{\text{cutoff}}} (k - K_E)^2 E^{1D}(k,t) \mathrm{d}k}\,.$$

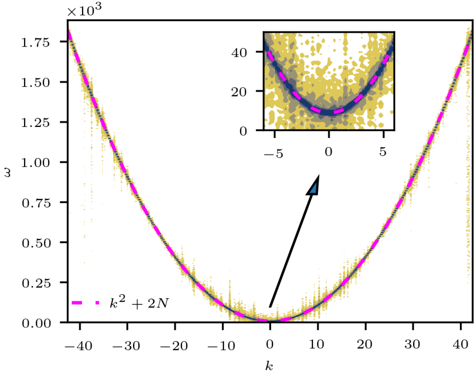
$E^{1D}(k,t) = k^2 n^{1D}(k,t)$
$N_c(t) = \int_0^{k_{\text{cutoff}}} n^{1D}(k,t) \mathrm{d}k\,,\; H_c(t) = \int_0^{k_{\text{cutoff}}} E^{1D}(k,t) \mathrm{d}k$

Evolution of 1D wave-action Spectrum for short time



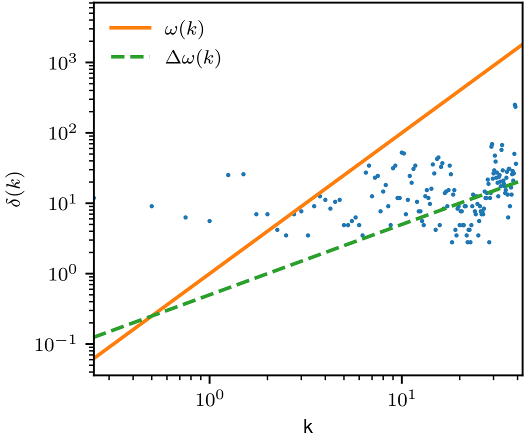
Dynamics of the centroids of $n^{1D}(k,t)$ and $E^{1D}(k,t)$ and their respective typical widths for short time.

Verifying WT assumptions for GPE settings



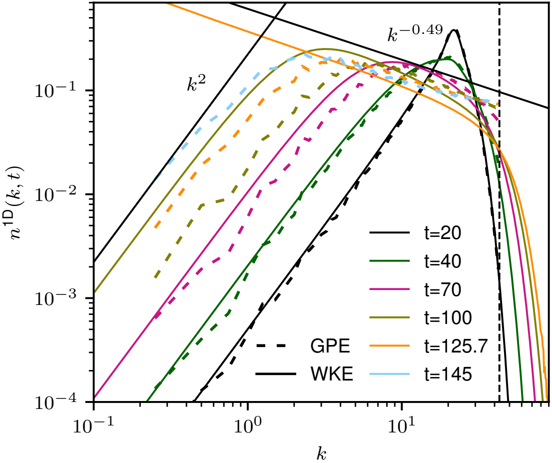
Spatio-temporal spectra density of $\psi(\boldsymbol{r}, t)$ over the time interval $[12, 18]$.

Verifying WT assumptions for GPE settings



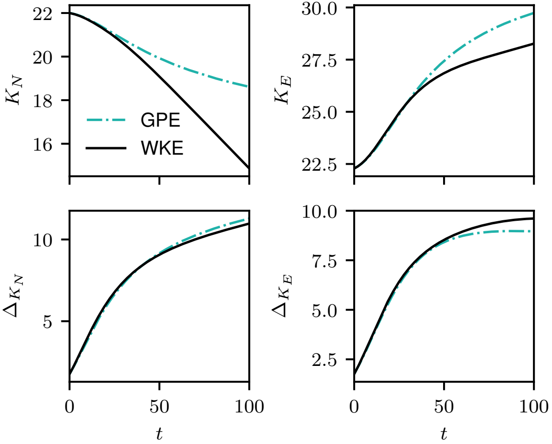
Frequency broadening $\delta(k)$ (blue points) vs. k obtained from the spatial-temporal spectral density over the time interval $[12, 18]$. $\omega(k) = k^2$, $\Delta\omega(k) = 2k\Delta k$

Evolution of 1D wave-action Spectrum for long time



Time evolution of $n^{1D}(k,t)$ for long time. $T_{kni} = 15$

Evolution of 1D wave-action Spectrum for long time



Dynamics of the centroids of $n^{1D}(k,t)$ and $E^{1D}(k,t)$ and their respective typical widths for long time.

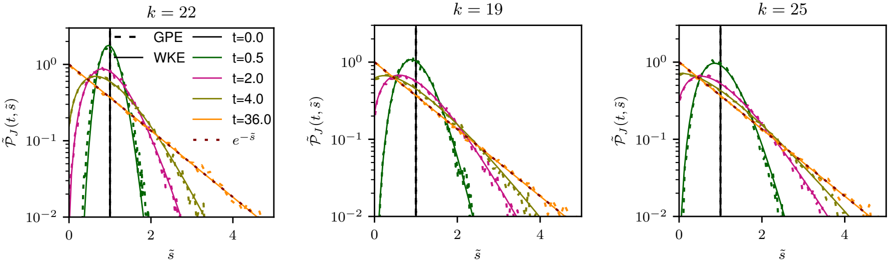
PDF and cumulants of wave-action

Theoretical prediction of PDF by [Choi et al., 2017(Journal of physics A)] :

$$\mathcal{P}_J(t,s) = \frac{1}{\tilde{n}} e^{-\frac{s}{\tilde{n}} - a\tilde{n}}$$

$\tilde{n} = n(t) - Je^{-\int_0^t \gamma(t')dt'} I_0(2\sqrt{as})$, $J = n(0)$, $a = \frac{J}{\tilde{n}^2} e^{-\int_0^t \gamma(t')dt'}$

$I_0(x)$, zeroth modified Bessel function of the first kind



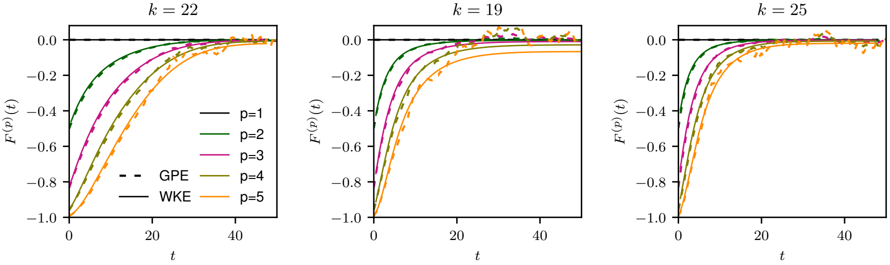
Time evolution of the normalized probability density functions $\tilde{P}_J(t,\tilde{s})$ of $n_{\mathbf{k}}$.

PDF and cumulants of wave-action

Cumulants : deviation from Gaussian distribution

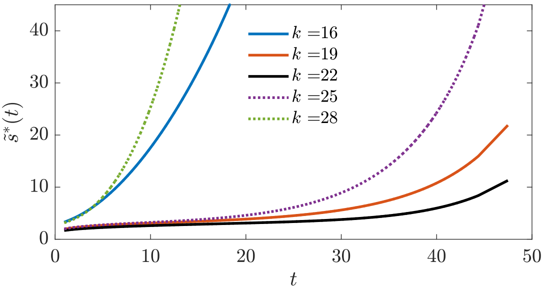
$$F^{(p)} = \frac{M^{(p)} - p!\,(M^{(1)})^p}{p!\,(M^{(1)})^p}\,, \quad M^{(p)} = \langle n_{\boldsymbol{k}}^p \rangle \qquad p = 1, 2, 3, \ldots$$

$F^{(p)} = 0$, Gaussian distribution



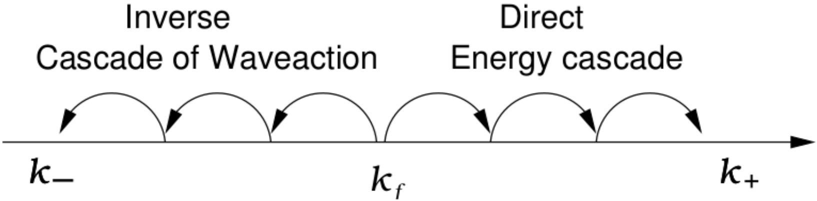
Time evolution of the relative cumulants $F^{(p)}(t)$ of $n_{\mathbf{k}}$.

PDF and cumulants of wave-action



Motion of the front of relative PDF for different values of k.

Stationary solutions



Stationary solutions

- Non-equilibrium stationary power-law, $n^{1D}(k) \propto k^{-\alpha}$
- direct cascade : $\alpha = -1$
  - Mystery -1.5 law for direct cascade in [Navon et al., 2016(Nature)]
  - log-correction by WWTT : $n^{1D}(k) \propto k^{-1} \ln^{-1/3}(\frac{k}{k_f})$, k_f forcing scale
- inverse cascade : $\alpha = -1/3$
  - dimension analysis : $n^{1D}(k) = ck^{-1/3}\zeta^{1/3}$, ζ flux of particles
  - prediction of the constant c by WWTT

$$N = 2\pi \int_0^\infty \omega^{1/2}\, n_\omega\, d\omega\,,\ \frac{\partial(2\pi\omega^{1/2} n_\omega)}{\partial t} = St_\omega$$

$$\frac{\partial(2\pi\omega^{1/2} n_\omega)}{\partial t} + \frac{\partial \zeta_\omega}{\partial \omega} = 0\,,\ \zeta_\omega = \int_0^\omega St_\omega d\omega$$

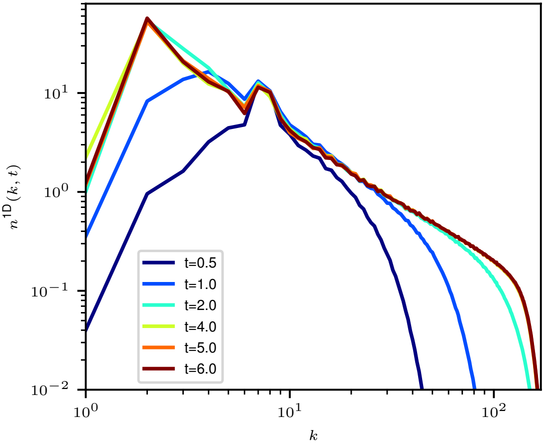
$$n_\omega = A\omega^\nu\,,\ St_\omega = 8\pi^4 A^3 \omega^{3\nu+5/3} I(x)\,,\ x = -7/2 - 3\nu$$

$$I(x) = \int S(1, q_1, q_2, q_3)\delta(1 + q_1 - q_2 - q_3)(q_1 q_2 q_3)^{-x/3-7/6}(1 + q_1^x - q_2^x - q_3^x)dq_1 dq_2 dq_3$$

$$\nu = -7/6\,,\ n_\omega = (\frac{\zeta_0}{8\pi^4 I'(0)})^{1/3}\omega^{-7/6}$$

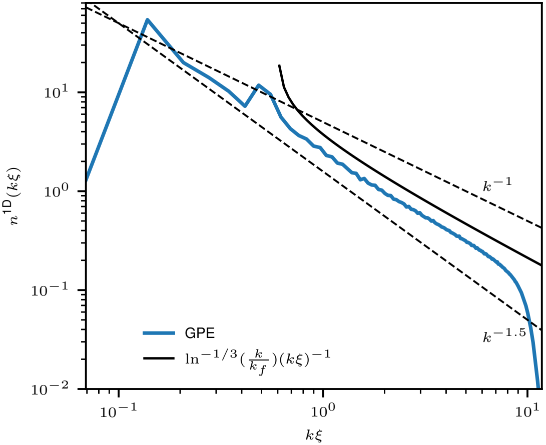
$$c = 2\left(\frac{1}{\pi I'(0)}\right)^{1/3}\,,\ I'(0) \approx -3.19\,,\ c \approx -0.928$$

Simulation results for direct cascade



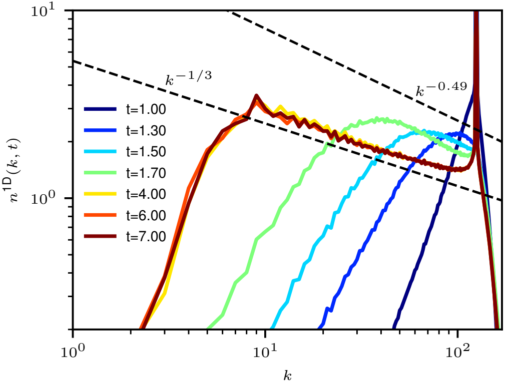
Time evolution of wave-action spectra by GPE

Simulation results for direct cascade



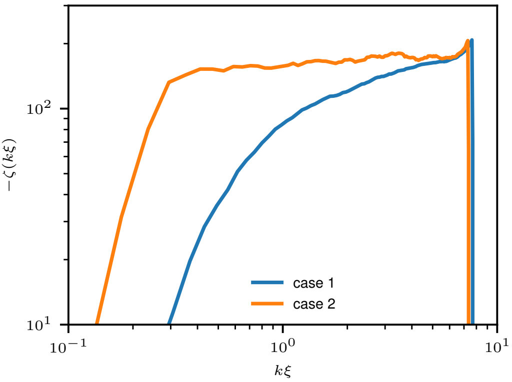
Stationary wave-action spectrum by GPE. healing length : $\xi = \frac{1}{\sqrt{N}}$

Simulation results for inverse cascade



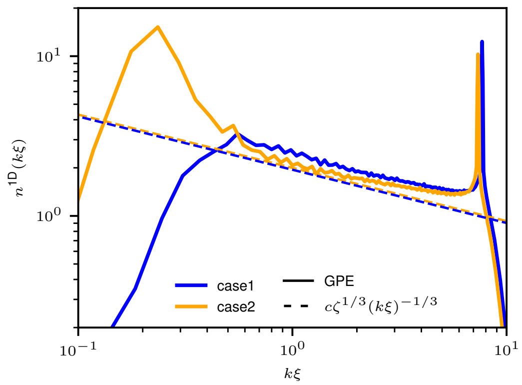
Time evolution of wave-action spectra by GPE

Simulation results for inverse cascade



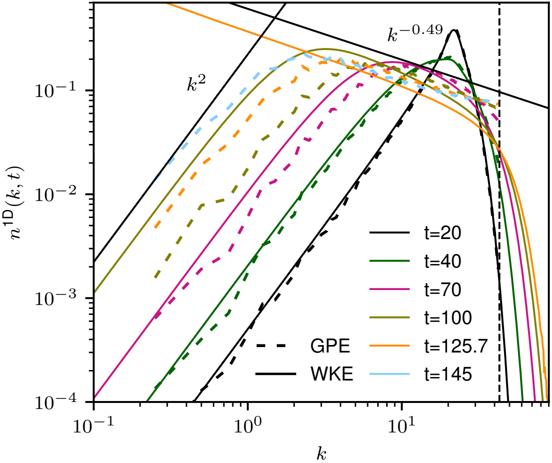
Fluxes of the stationary solution for different dissipation at low k by GPE.

Simulation results for inverse cascade



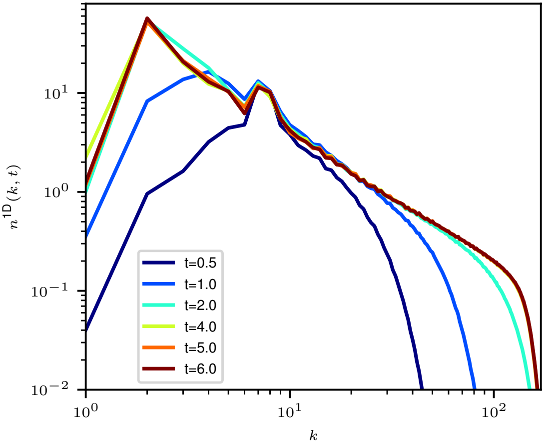
Stationary wave-action spectra. Comparison between GPE and theoretical prediction.

Self-similar solutions



Example of self-similar solution, free decay case.

Self-similar solutions

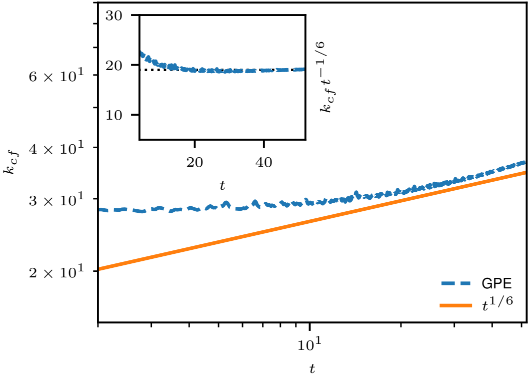


Example of self-similar solution, forcing case.

Self-similar solutions

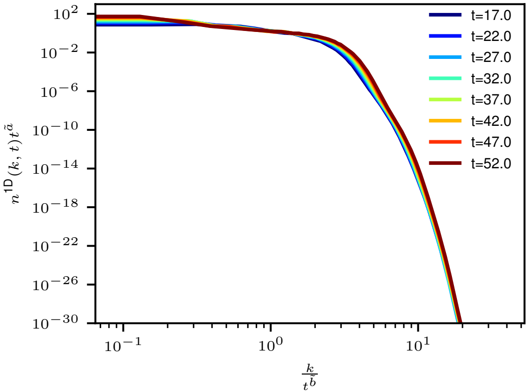
- suppose $n_\omega = t^{-a} F(\eta)\,, \quad \eta = \frac{\omega}{t^b}$
  substitute to WKE, $a - b = \frac{1}{2}$
- for free decay case, consider the conservation of energy
  $H = 2\pi \int\limits_0^\infty \omega^{3/2} n_\omega \, d\omega,$
  we get $a - \frac{5}{2} b = 0$
- for forcing case, suppose $H \propto t$, we get $a - \frac{5}{2} b = -1$
- transfer to k space, $n^{1D}(k, t) = t^{-\tilde{a}} f\left(\frac{k}{t^{\tilde{b}}}\right)$
  for free decay case $\tilde{a} = 1/2$ and $\tilde{b} = 1/6$
  for forcing case $\tilde{a} = 1/2$ and $\tilde{b} = 1/2$

Simulation results for free decay case.



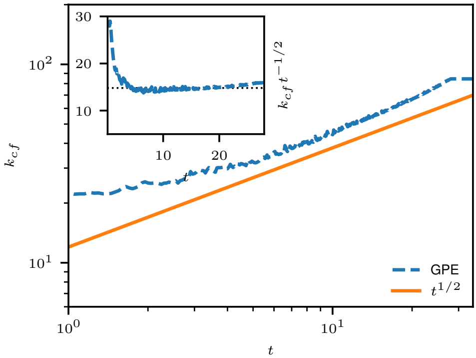
Time evolution of the front of wave-action spectra.

Simulation results for free decay case.



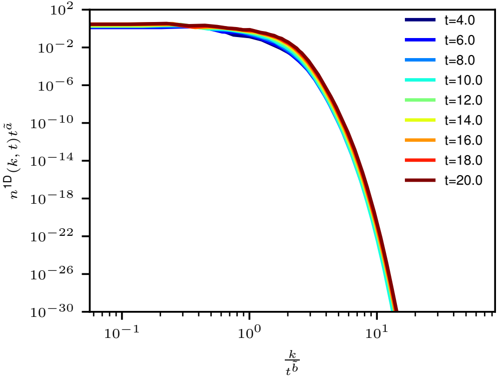
Time evolution of $f\left(\frac{k}{t^{\tilde{b}}}\right)$

Simulation results for forcing case



Time evolution of the front of wave-action spectra. [Navon et al., 2019(Nature)]

Simulation results for forcing case



Time evolution of $f\left(\frac{k}{t^{\tilde{b}}}\right)$.

Conclusions

- We correct the pre-factor of WKE
- The temporal evolution of GPE data is accurately predicted by the WKE, with no adjustable parameters
- For the first time, comparative analysis of PDF and cumulants for GPE and WKE has been done
- The characteristic times of wave-action spectra and PDF are of the same order
- We explain the mystery scaling of stationary direct cascade obtained by experiments
- We achieve the stationary scaling for inverse cascade and find out the flux constant for the first time
- The second kind of self-similar solutions are solved with WWTT, for both free decay case and forcing case

Thanks for your attention !

Q&A