# Minimal Modelling of the Quasi-Periodic Behaviour of Turbulence

Preprint: “Intrinsic Periodicity of Turbulence” (arXiv.2112.03417)

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ÉCOLE CENTRALE LYON

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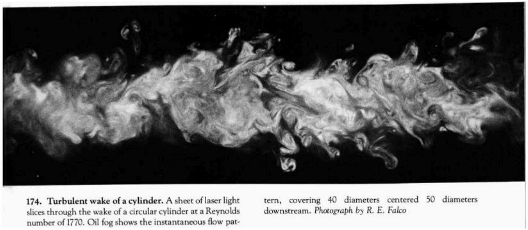
# Intrinsic Periodicity of Turbulence

Experiment in Goto lab @ Osaka Univ., courtesy of H. Ohyama.

(a) Low $\mathrm{Re}$



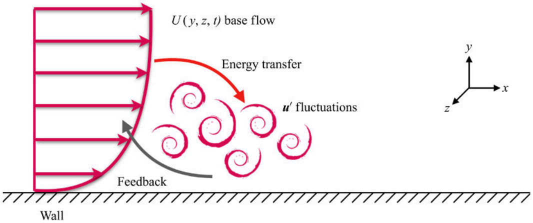

M. Van Dike, "*An Album of Fluid Motion*" (1982).

(b) High $\mathrm{Re}$

*... at very high Reynolds number, there appears a tendency to restore the symmetries in a statistical sense ...*  *U. Frisch, "Turbulence" (1995)*

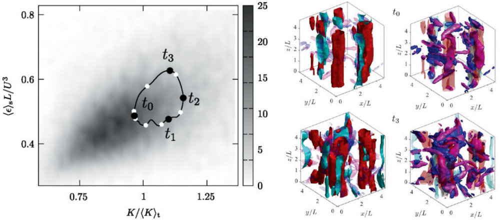
# Self-sustaining process

F. Waleffe, *Phys. Fluids* (1997), A. L.-Durán, *et al.*, *J. Fluid Mech.* (2021)



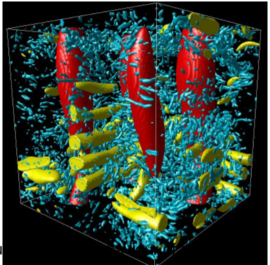

# Unstable periodic orbit

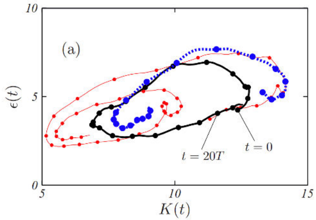
L. v. Veen, A. Vela-Martín, & G. Kawahara, *Phys. Rev. Lett.* (2019)



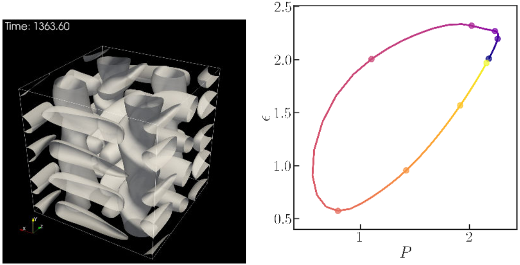

# Vortex stretching & energy cascade

S. Goto, Y. Saito, & G. Kawahara, *Phys. Rev. Fluids* (2017)

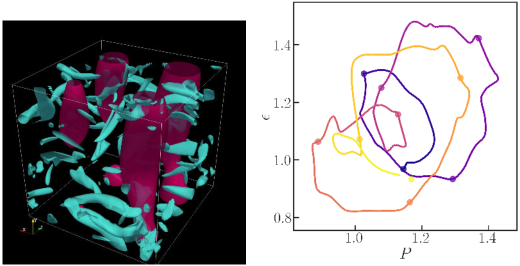




# Quasi-Periodic Behaviour of Box Turbulence: DNS Set-Up




(a) 3D Periodic Flow (3DPF)



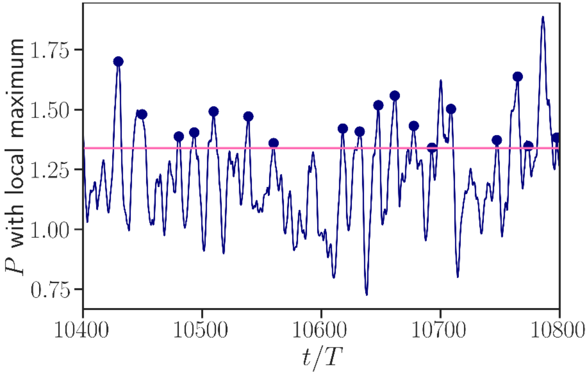

(b) Turbulent flow

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} p + \nu\nabla^2\boldsymbol{u} + \underbrace{(-\sin x\cos y, \cos x \sin y, 0)}_{\boldsymbol{f}}$$

- In $(2\pi)^3$ periodic box and steady forcing $\boldsymbol{f}$.
- There are Quasi-Periodic Behaviour (QPB).

# Quasi-Periodic Behaviour of Box Turbulence: Phase-Averaging

1. Pick local maximum of $P$.
   - Thresholds on magnitude & minimum gap.
2. Overlap time series.
   - $x$ axis: $t - t_{\text{local maximum}}$.
   - Normalise by local maximum.
3. Compute average of them.
   - Same procedure for $\epsilon$, but with local maximum of $P$.




QPB is a robust feature irrespective to $\mathrm{Re}$.
What would be the simplest possible expression of such behaviour?

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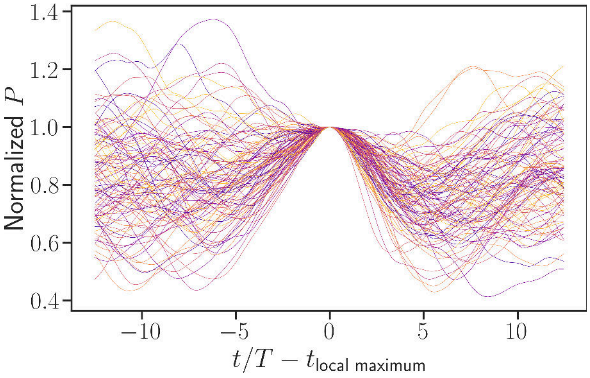


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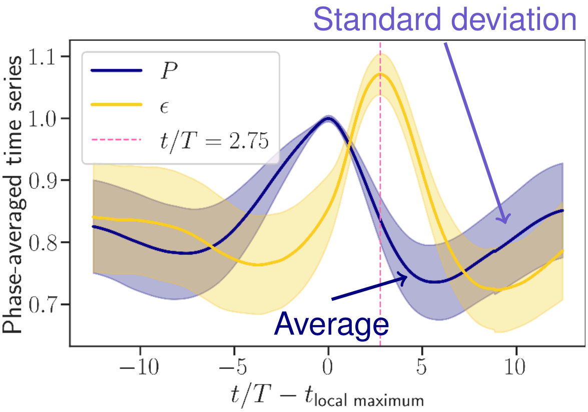


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# Phase-Averaging

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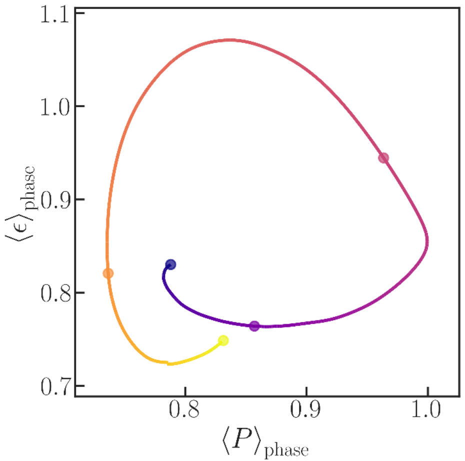
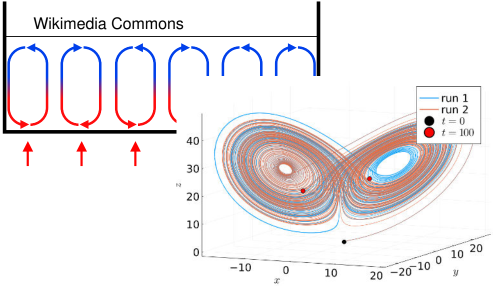



QPB is a robust feature irrespective to $\mathrm{Re}$.
What would be the simplest possible expression of such behaviour?

# Background of the Model

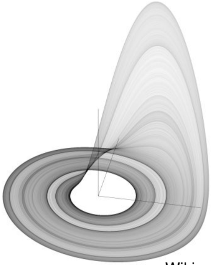
## Lorenz model Lorenz (1963)

$$
\begin{cases}
\mathrm{d}_t X = \qquad\quad -\sigma X + \sigma Y, \\
\mathrm{d}_t Y = -XZ + rX - Y, \\
\mathrm{d}_t Z = +XY \qquad\quad - bZ.
\end{cases}
$$




## Rössler model Rössler (1976)

$$
\begin{cases}
\mathrm{d}_t X = -Y - Z \\
\mathrm{d}_t Y = +X + aY \\
\mathrm{d}_t Z = +b + Z(X - c)
\end{cases}
$$



Wikipedia

# Background of the Model

## Turbulence shell model   L'vov (1998)

$$\mathrm{d}_t u_n = \mathrm{i}B_n - \nu k_n^2 u_i + f_n,$$

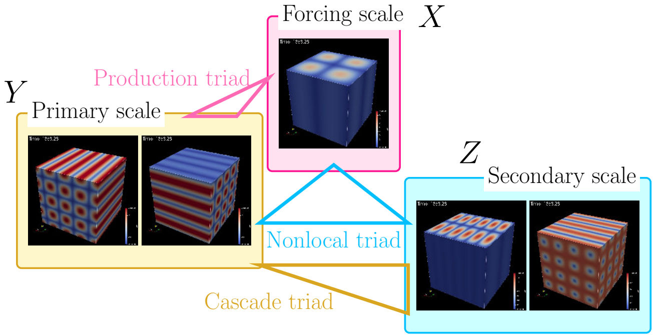
$$B_n = \left( k_{n+1} u_{n+2} u_{n+1}^* - \frac{1}{2} k_n u_{n+1} u_{n-1}^* + \frac{1}{2} k_{n-1} u_{n-1} u_{n-2} \right)$$

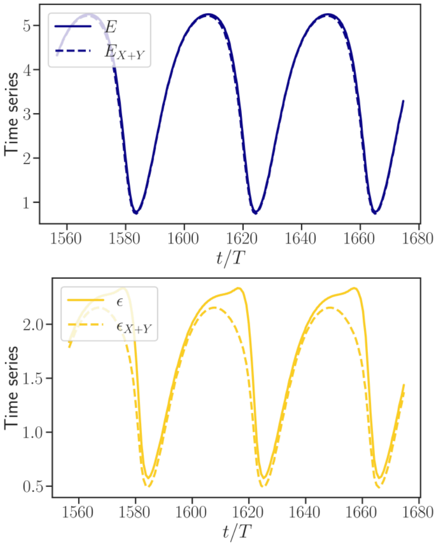
- Complex shell variables $u_n$ and their scale-local interactions.

## Discrete Navier-Stokes equation   Kraichnan (1958, 1988)

$$[\partial_t + \nu_i] q_i = \sum_{j,m} A_{ijm} q_j q_m + f_i$$

- Fourier representation of the individual mode $q_i$.

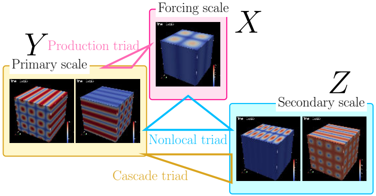


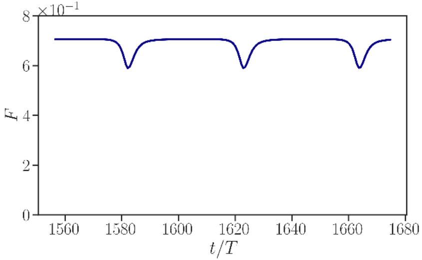

1. Forcing scale: $X$
   - Columnar vortices.
2. Primary scale: $Y$
   - 1. and 2. dominates energy.
3. Secondary scale: $Z$
   - The rest of the modes.

$$\begin{cases} \mathrm{d}_t X = -A_1 Y^2 + A_3 YZ - \mathrm{Re}^{-1} K_X^2 X + F, \\ \mathrm{d}_t Y = +A_1 XY - A_2 Z^2 + A_4 XZ - \mathrm{Re}^{-1} K_Y^2 Y, \\ \mathrm{d}_t Z = +A_2 YZ - (A_3 + A_4) XY - \mathrm{Re}^{-1} K_Z^2 Z. \end{cases}$$

- Three triads represent energy transfer:
  - "Production" $A_1$ : $X \to Y$.
  - "Cascade" $A_2$ : $Y \to Z$.
  - "Nonlocal" $A_3, A_4$: between $X, Y$, and $Z$.
- Viscous $\mathrm{Re}^{-1} K_\alpha^2$ and steady forcing $F$ terms.



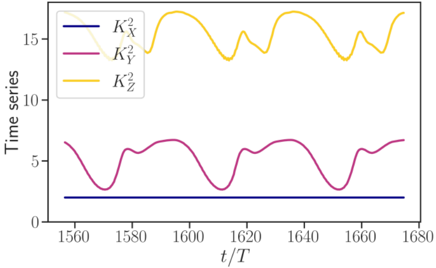
## Forcing coefficient $F$




$$F = \langle \boldsymbol{f} \cdot \boldsymbol{u} \rangle / \sqrt{\langle |\boldsymbol{u}_X|^2 \rangle}$$

- Balance between energy input rate and characteristic velocity.

## Scaling factors $K_\alpha^2$




$$K_\alpha^2(t) = \epsilon_\alpha(t)/2\nu E_\alpha(t)$$

- Balance between energy and energy dissipation rate.

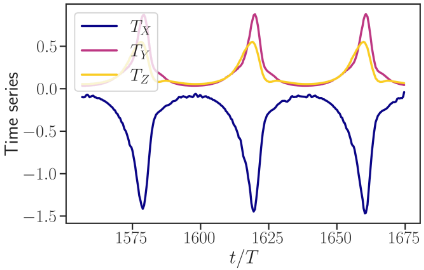
## Energy transfer terms $T_\alpha$ and their coefficients $A_i$

- Energy equations

$$\begin{cases} \mathrm{d}_t E_X = T_X - \epsilon_X + P, \\ \mathrm{d}_t E_Y = T_Y - \epsilon_Y, \\ \mathrm{d}_t E_Z = T_Z - \epsilon_Z, \end{cases}$$

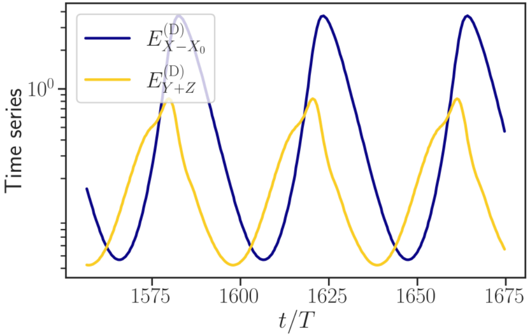
- Energy transfer terms

$$\begin{cases} T_X = -A_1 XY^2 \qquad\qquad + A_3 XYZ \\ T_Y = +A_1 XY^2 - A_2 YZ^2 + A_4 XYZ \\ T_Z = \qquad\qquad + A_2 YZ^2 - (A_3 + A_4) XYZ \end{cases}$$

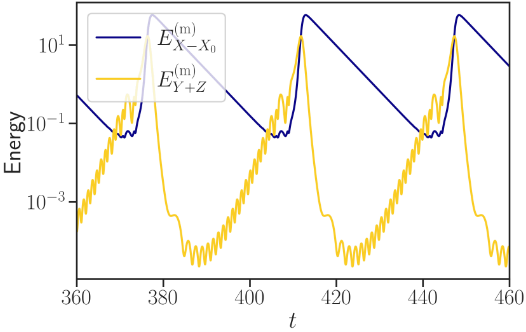



- Neglect $A_3,\ A_4$ to determine $A_1,\ A_2$.

## DNS




## Model

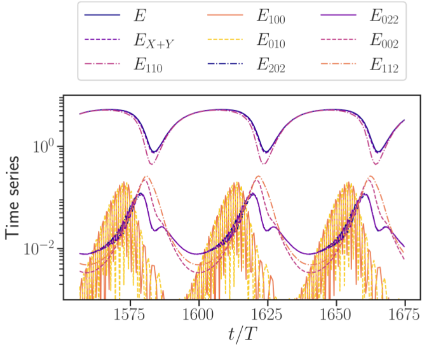



$$\begin{cases} E^{(D)}_{X-X_0} \equiv \left\langle \left| \boldsymbol{u}_X - \boldsymbol{f}/2\nu|\boldsymbol{k}_f|^2 \right|^2 \right\rangle /2, \\ E^{(D)}_{Y+Z} \equiv \left\langle |\boldsymbol{u}_Y|^2 \right\rangle /2 + \left\langle |\boldsymbol{u}_Z|^2 \right\rangle /2. \end{cases}$$

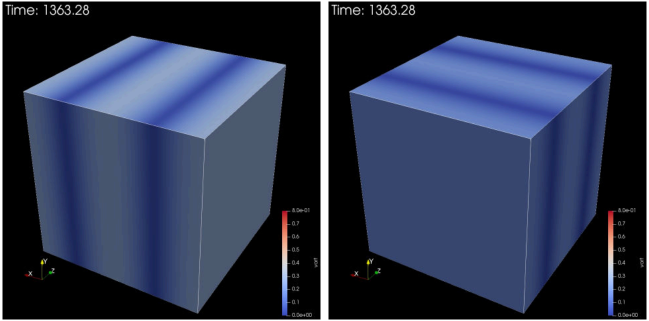
$$\begin{cases} E^{(m)}_{X-X_0} \equiv \left( X - F\,\mathrm{Re}/K_X^2 \right)^2 /2, \\ E^{(m)}_{Y+Z} \equiv Y^2/2 + Z^2/2. \end{cases}$$

- QPB $\leftrightarrow$ predator-prey dynamics?

# Three-Equation Model of QPB: “Fast Oscillations” in DNS

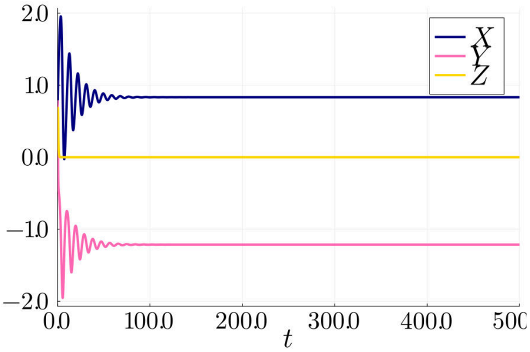



(a) Energy of specific modes

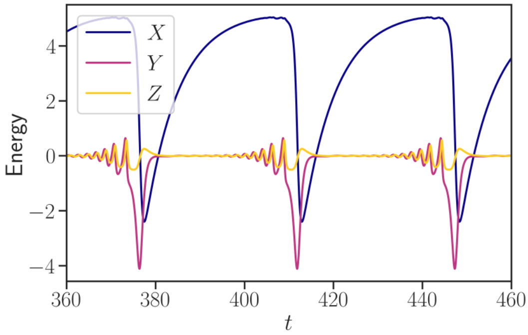



(b) $\boldsymbol{\omega}$ of $\boldsymbol{k} = (1, 0, 0)$ and $(0, 1, 0)$ modes

- “Fast oscillations” appear in DNS as well,
  but they are compensated by their symmetric modes.



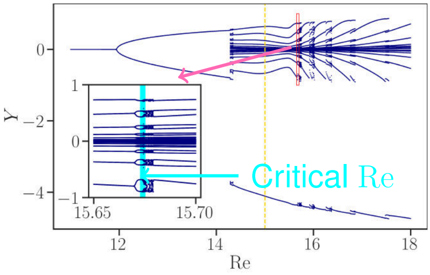

(a) $A_3 = A_4 = 0$



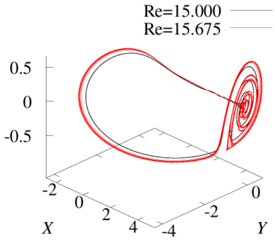

(b) $A_3 = 0.5,\ A_4 = -1$

- We do not observe QPB without non-local interactions.
  - Non-local interactions are mandatory to maintain QPB.

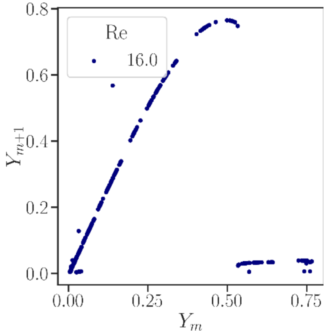
## Supercritical transition with $A_3 = 0.5,\ A_4 = -1$




(a) Bifurcation diagram

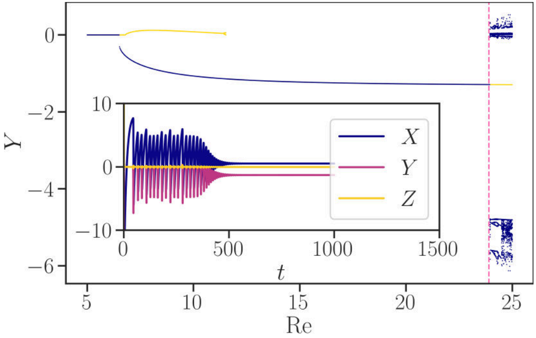



(b) Solution orbits

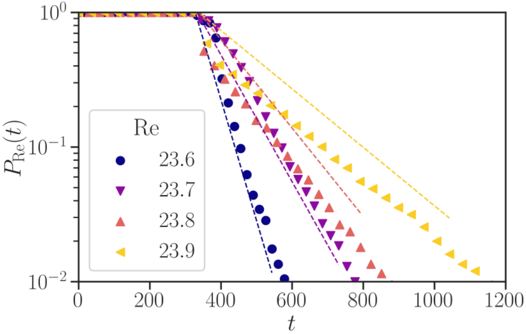



(c) Return map

## Subcritical transition with $A_3 = 0.5,\ A_4 = -0.63$




(a) Bifurcation diagram



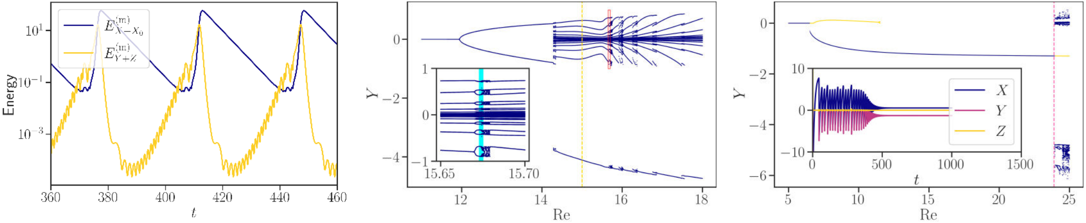

(b) Survival probability

- Subcritical transition between steady and transient chaos.
- "Survival time" follows exponential decay → "sudden relaminarisation".

# Objective

- Reproduce robust Quasi Periodic Behaviour (QPB) in turbulence by a simplest possible model.

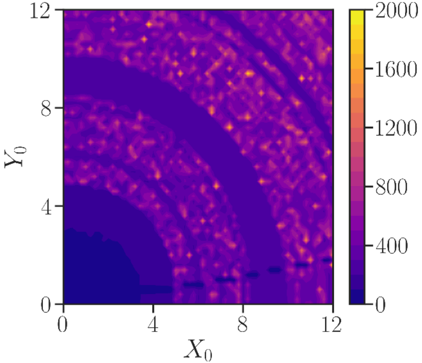
# Results



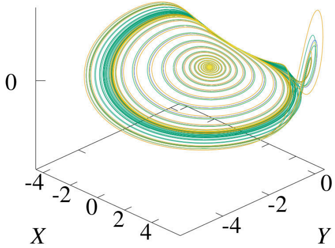

(a) QPB (b) Supercritival (c) Subcritival

- Propose a three-equation model retains N-S structures.
- It reproduces several key dynamics of turbulent flows.
- Preprint: “Intrinsic Periodicity of Turbulence” (arXiv.2112.03417).

# Perspectives: detail of the subcritical transitions



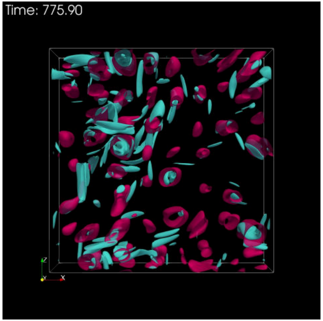

(a) Survival time map



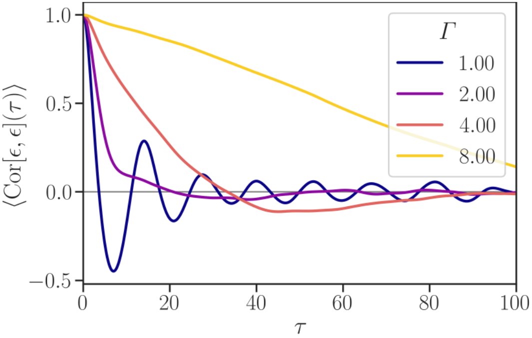

(b) “Sudden relaminarisation”

- How does the model help understanding the subcritical transition?
- “Edge tracking” the transient chaos.

# Perspectives: importance of the "space-locality"




(a) Turbulence $\in (8\pi, 8\pi, 2\pi)$




(b) Two-time autocorrelations of $\epsilon$

- How does the "space-locality" affect QPB?
- Investigate "$(2n)^2$ vortices flow" expanded horizontally.