Asymptotic reduction for rapidly rotating convection

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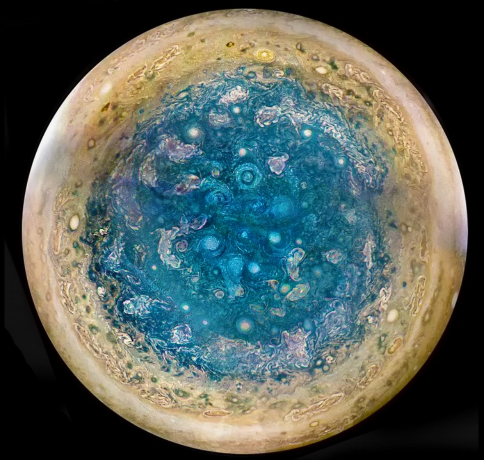
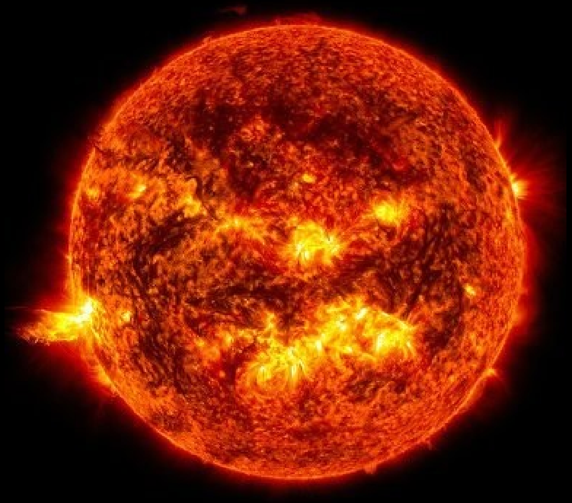
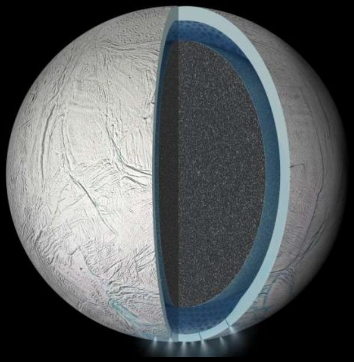
Turbulent transport in nature ?







Geo– and astrophysical flows

Outer layers : Oceans, Atmospheres
Planetary and stellar interiors

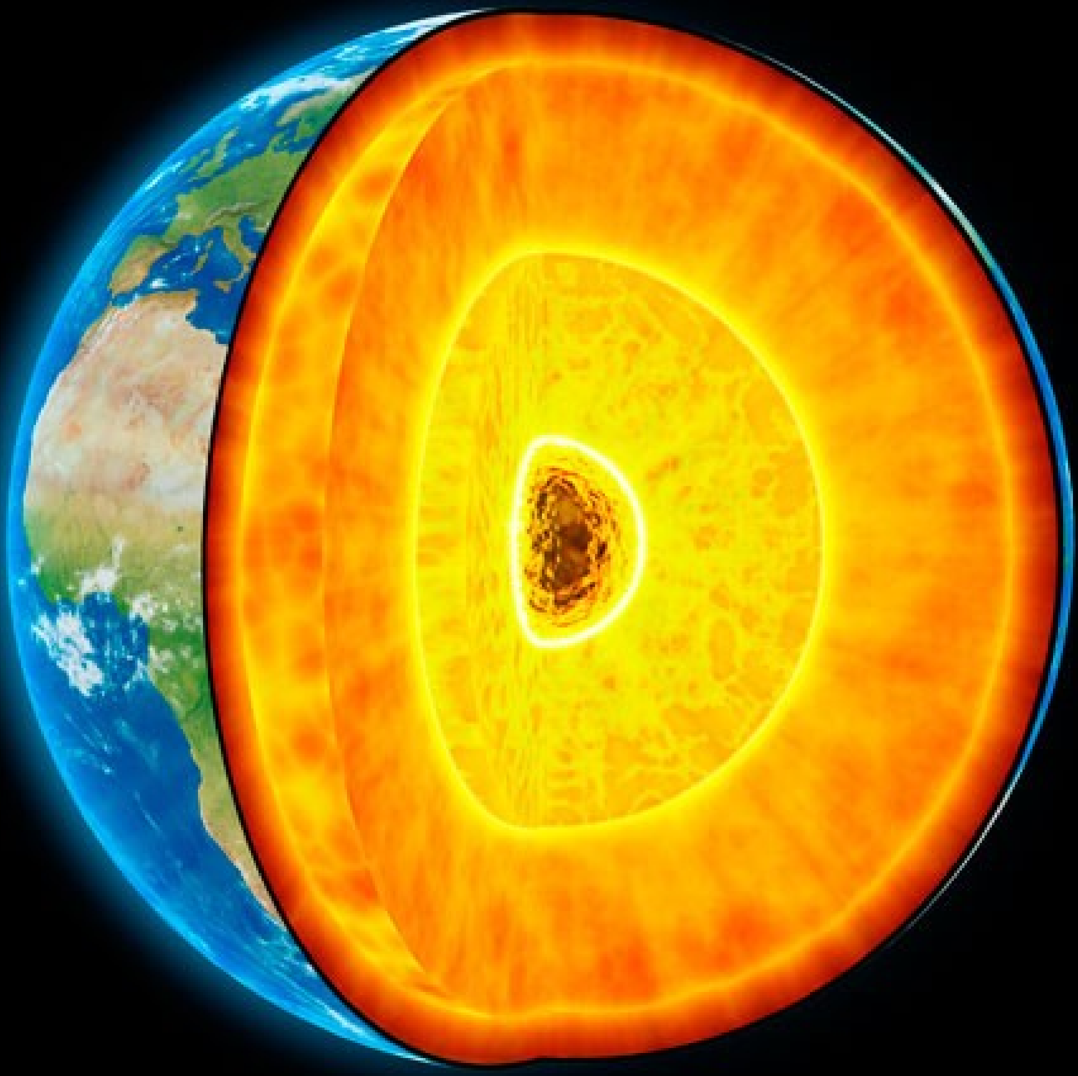
– ***Global rotation*** (low Rossby and Ekman numbers)

– Temperature and/or compositional inhomogeneities : ***Stratification***

– Phenomenology : anisotropy, inertio–gravity waves, Ekman layers, etc.

– Challenges : **Turbulence** and **rapid** rotation

Turbulent transport in nature ?



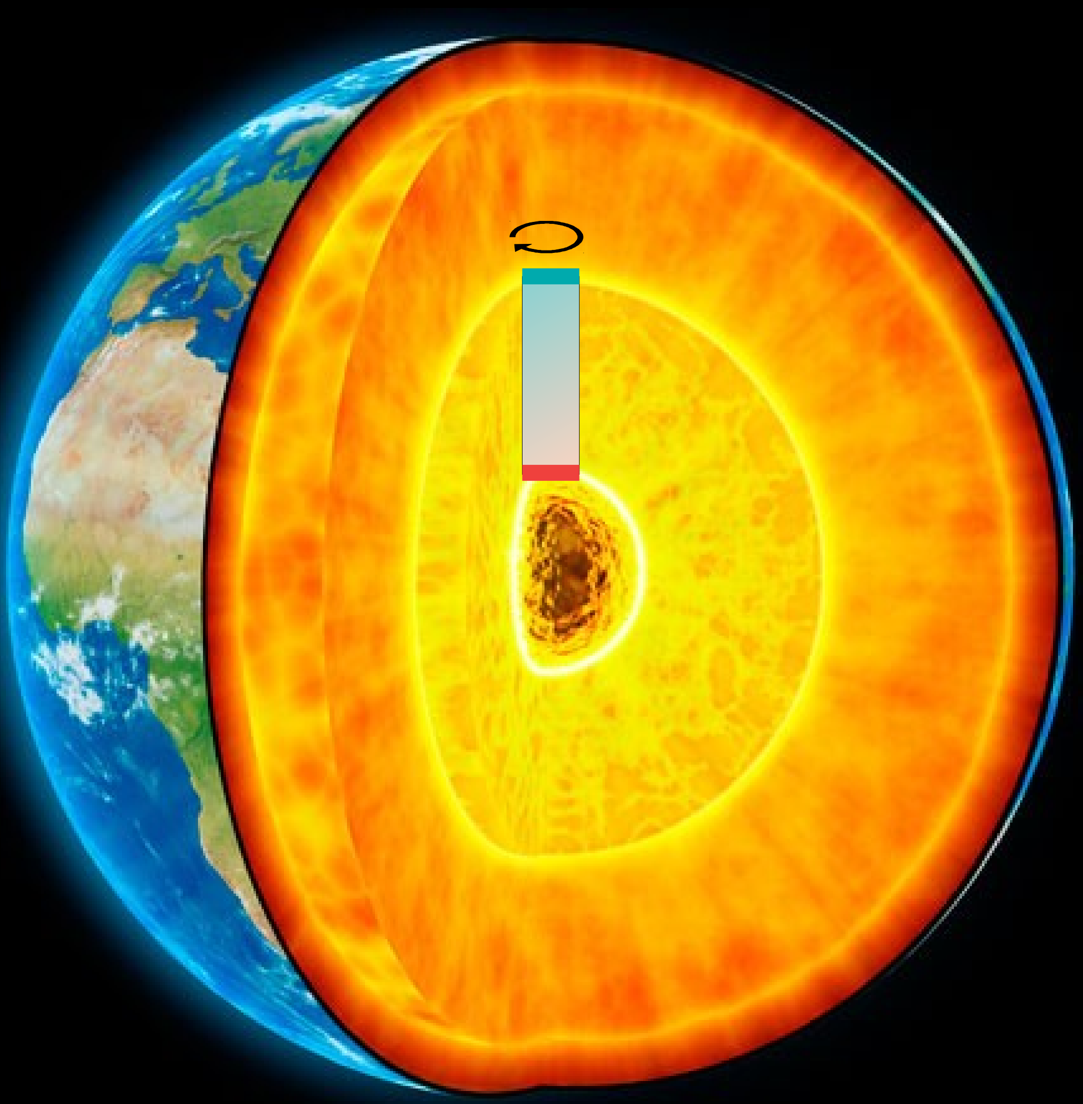
The Earth' outer core

$$E = \frac{\nu}{2\Omega H} \sim 10^{-15}$$

– Strategy : identify **power laws** for extrapolation

– Simplified and idealized models
Boussinesq fluids
Local geometry

Turbulent transport in nature ?



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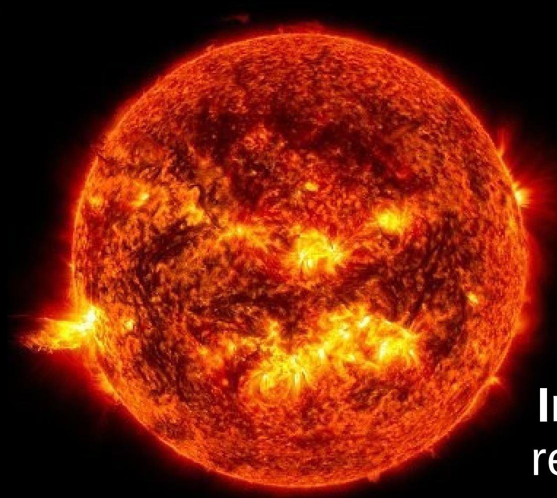
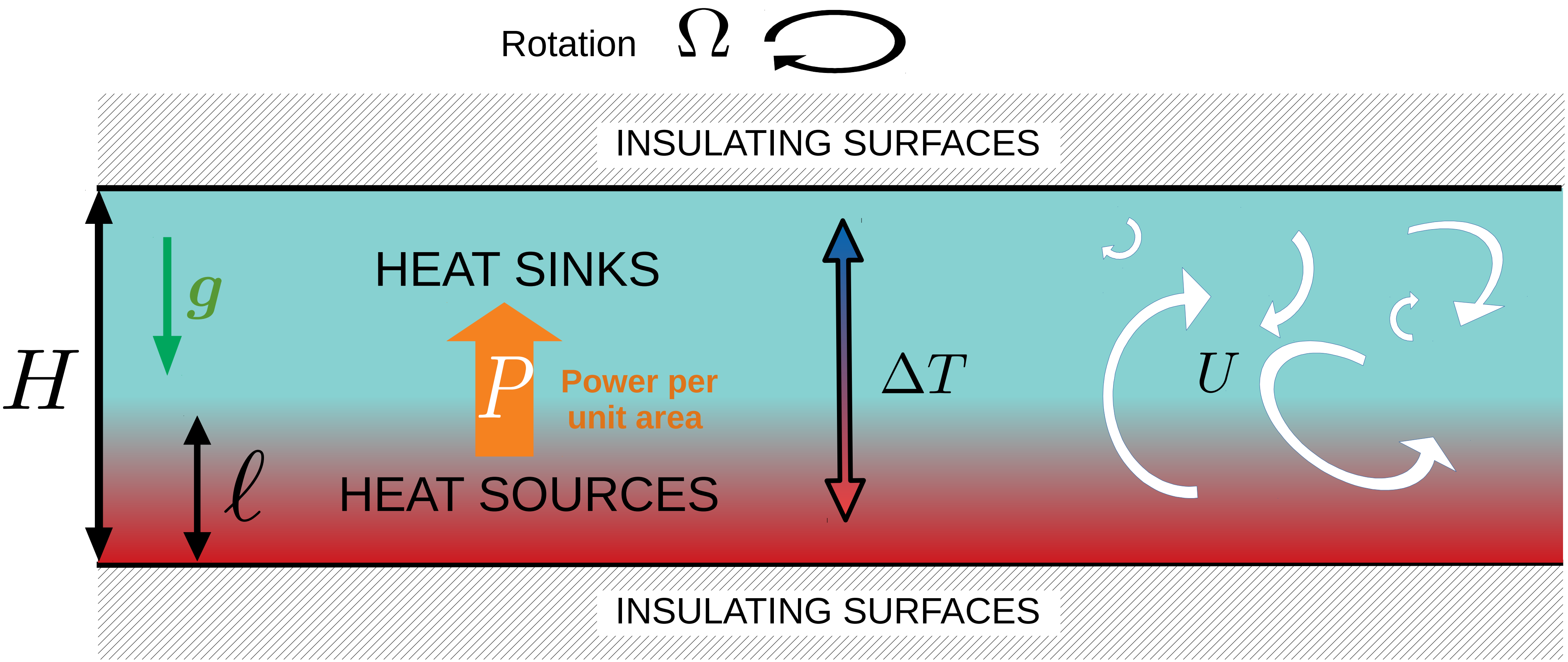
Rotating Rayleigh–Benard ? Wall heating



Internal heating
relevant for stars !

Internal Heating : a simple model for convection

Heat input directly in the bulk of the flow



Transport parametrization in these systems ???

$$P = f_1(\Delta T, \nu, \kappa, H, \Omega...)???$$
$$U = f_2(\Delta T, \nu, \kappa, H, \Omega...)???$$

Internal Heating : a simple model for convection

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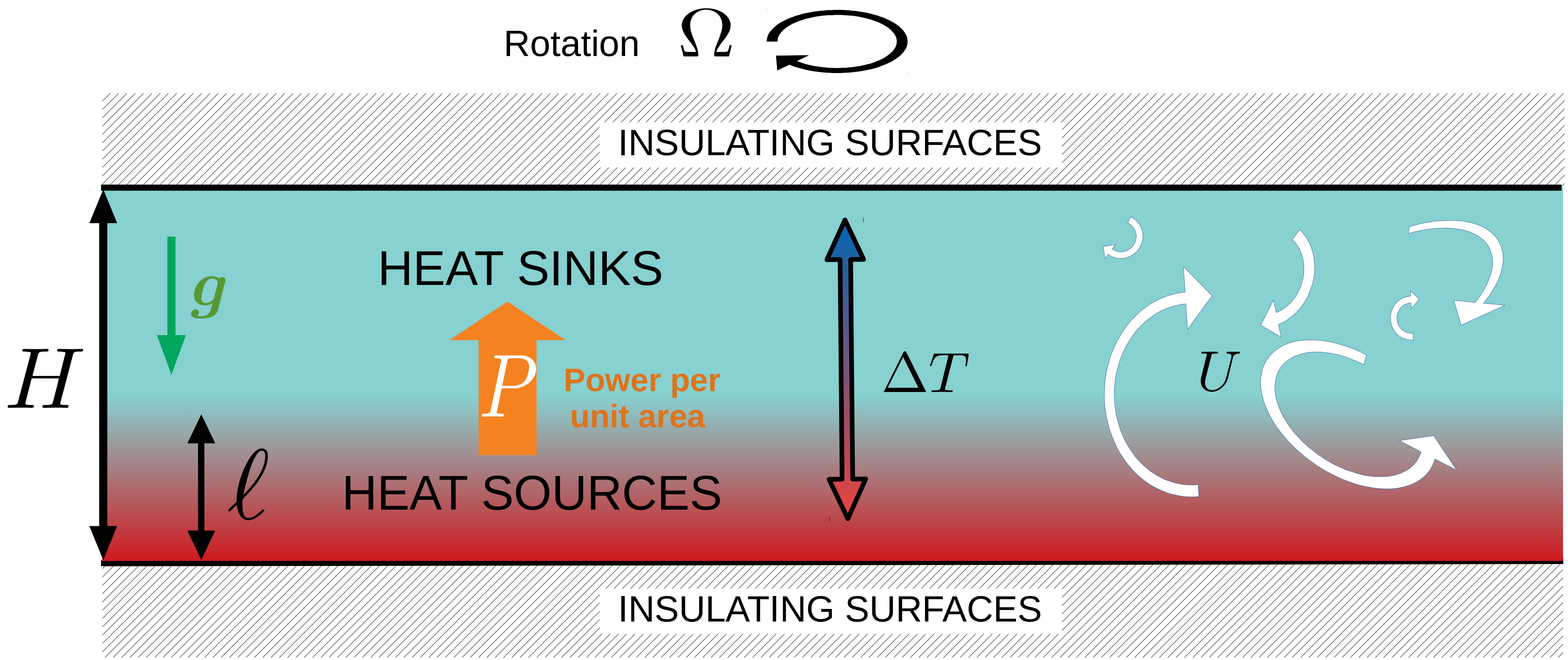


Dimensionless control parameters :

$$\mathrm{Ra}_Q = \frac{\alpha g P H^4}{\rho_0 C \kappa^2 \nu}$$

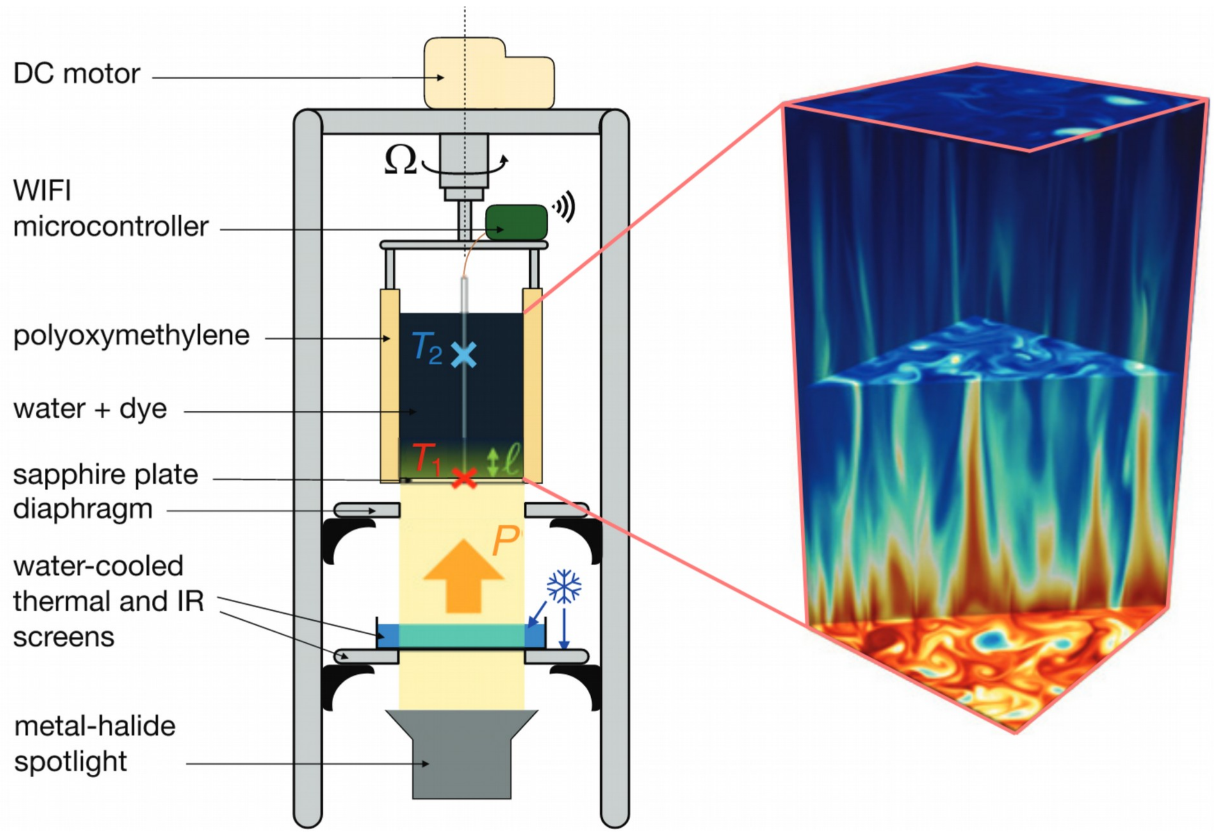
$$\mathrm{Pr} = \nu/\kappa; \quad \mathrm{E} = \nu/2\Omega H^2$$

Flow response :

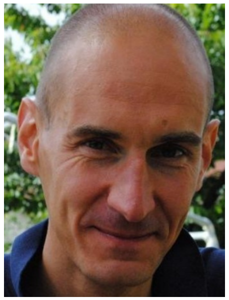
$$\mathrm{Nu} = \frac{PH}{\rho_0 C \kappa \Delta T}$$

$$\mathrm{Re} = UL/\nu$$

Internal Heating in the lab




Basile
Gallet


Sébastien
Aumaître


Vincent
Bouillaut

Exp. CEA – SPEC

Experiments :

Lepot et al., PNAS 2018
Bouillaut et al., JFM 2019
Bouillaut et al., PNAS 2021

Numerics :

Miquel et al., PRF 2019
Miquel et al., JFM 2020

at LMFA : Creyssels, JFM 2021

Turbulent (diffusivity – free) scalings

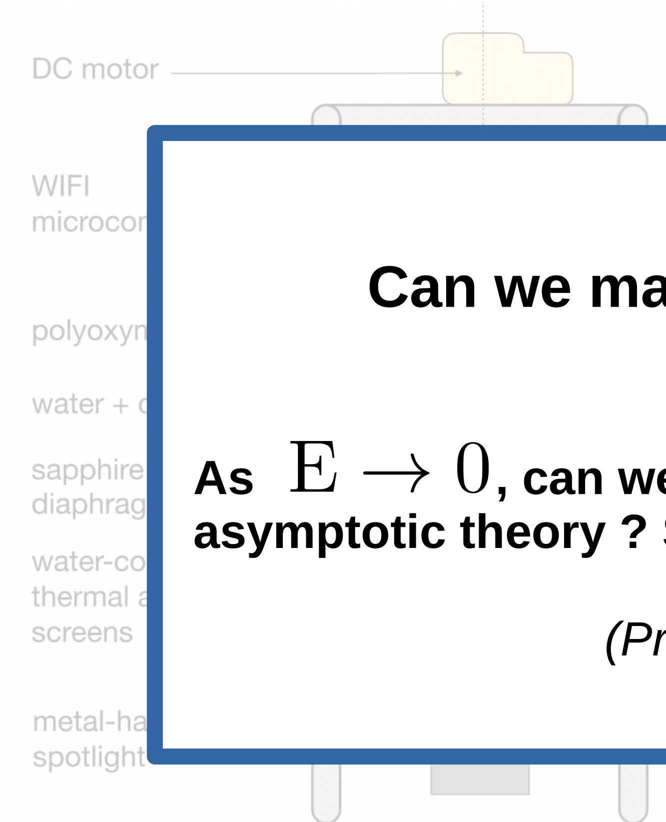
$$\mathrm{Nu} \sim \sqrt{\mathrm{Ra} \times \mathrm{Pr}}$$

Heat flux — Temperature difference — Diffusivity ratio

With rotation

$$\mathrm{Nu} \sim \left(\mathrm{Ra}_Q \mathrm{E}^{4/3}\right)^{3/5} / \mathrm{Pr}^{1/5}$$

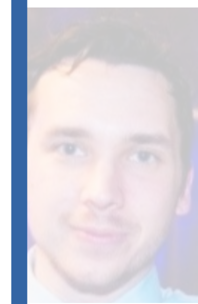
Internal Heating : the $\mathrm{E} \to 0$ limit



Can we map out the different regimes and their transitions ?

As $\mathrm{E} \to 0$, can we capture the leading order dynamics with an asymptotic theory ? Scaling and convergence of solutions ?

(Preliminary results in this talk, but still work in progress...)



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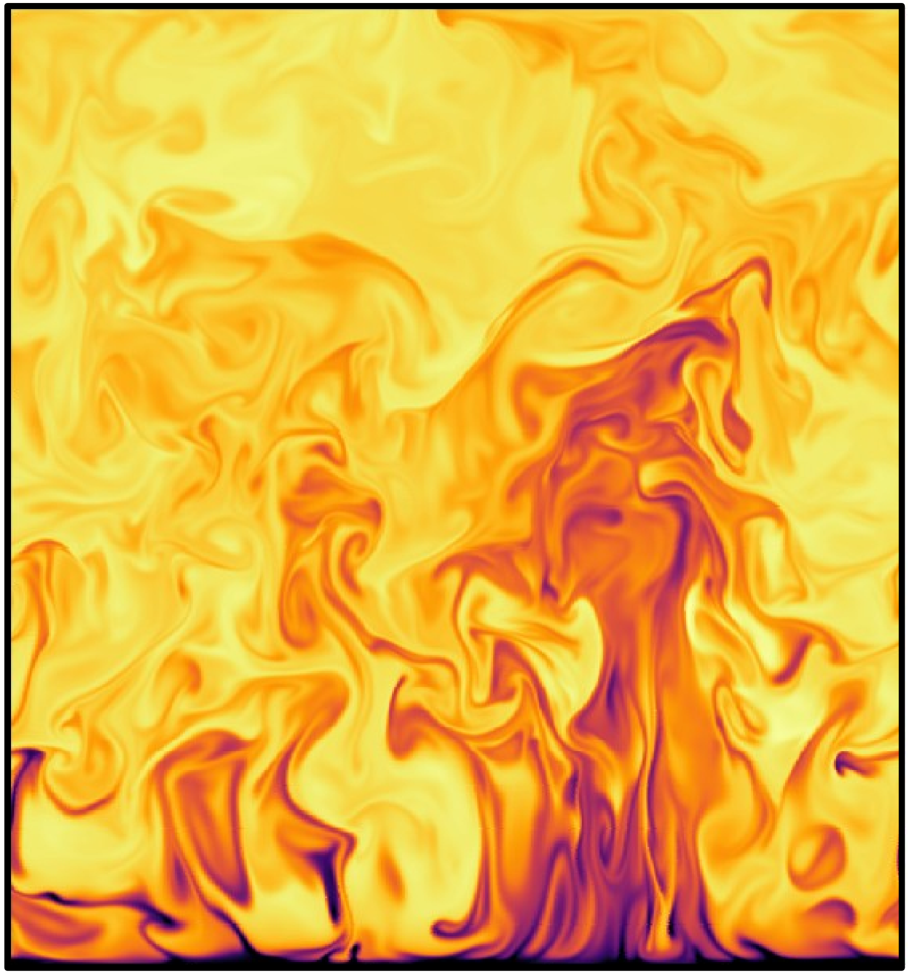
Coral — a parallel spectral solver for PDEs



A flexible framework for solving differential eqs.

– Geometry :
periodic plane layer and cylindrical shell

– Spectral decomposition:
Chebyshev Fourier Fourier

– Modularity and flexibility:
Equations entered as a simple text file
(no coding needed!)

– Scope :
Quadratic PDEs, i.e. advection-diffusion
and most flavours of Navier – Stokes
(stratified, rotating, convective, sheared, MHD, etc.)

– Fast and scalable:
Fortran 2003/2008, MPI

– Open source, git it !
`www.github.com/benMql/coral`
(Miquel, J. Open Source Soft. 2021)

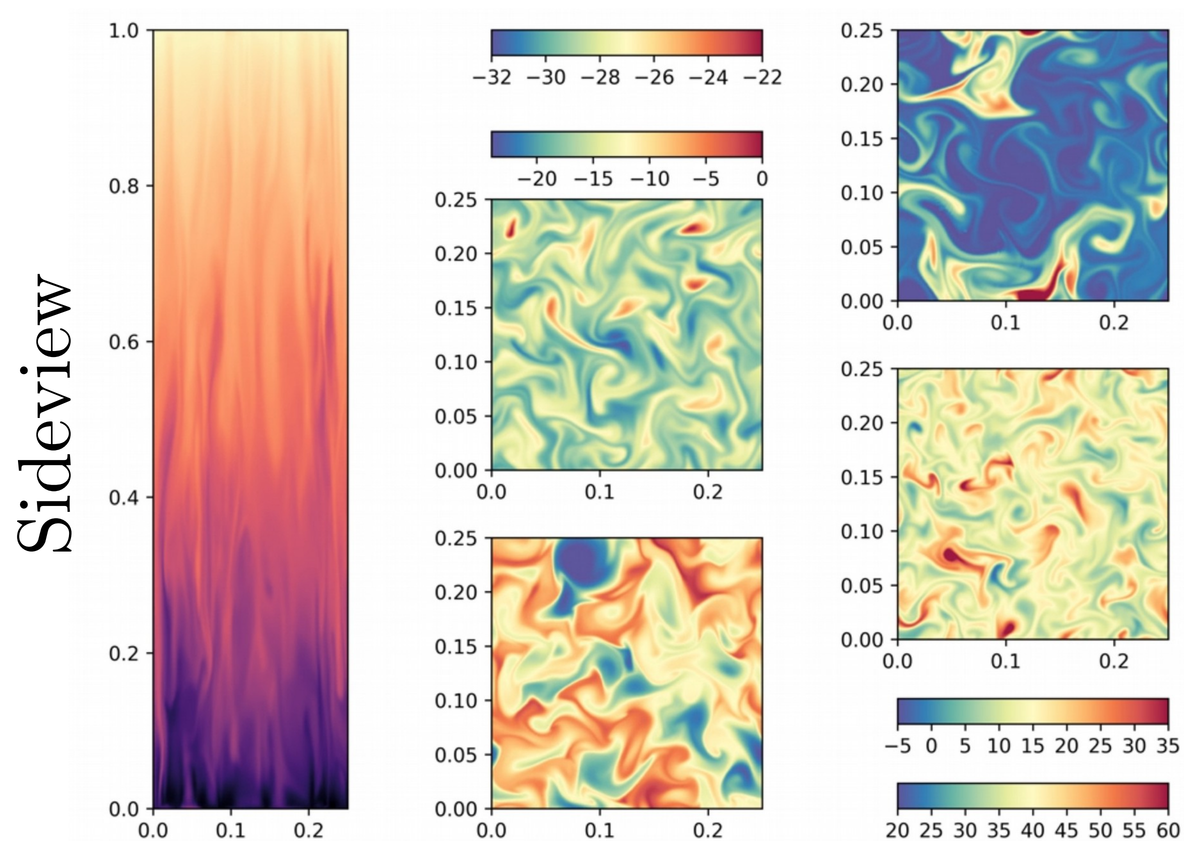
Numerical solutions

$$\frac{1}{\mathrm{Pr}} D_t u + \mathrm{E}^{-1} \hat{\boldsymbol{e}}_{\boldsymbol{z}} \times \boldsymbol{u} = \mathrm{E}^{-1} \boldsymbol{\nabla} p + \mathrm{Ra}_Q \Theta \hat{\boldsymbol{e}}_{\boldsymbol{z}} + \nabla^2 \boldsymbol{u}$$

advection Coriolis buoyancy viscosity

$$D_t \Theta = \nabla^2 \Theta + \left(\frac{1}{N_\ell} \exp\left(-z/\tilde{\ell}\right) - 1 \right)$$

advection diffusion Heat source and sink



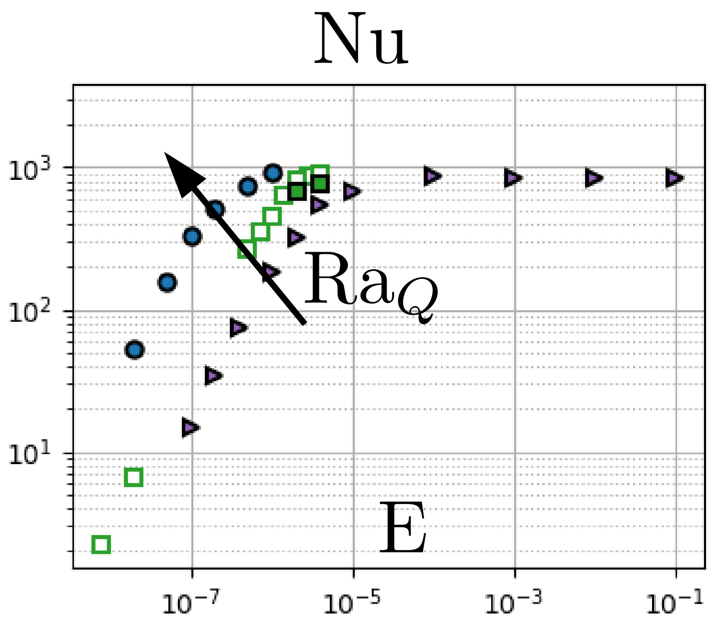
Visu : Temperature Θ

$$\mathrm{Ra}_Q = 1.2 \times 10^{13}$$
$$\mathrm{E} = 5 \times 10^{-7}$$
$$\mathrm{Pr} = 7$$
$$\tilde{\ell} = 0.024$$

$$z = 0,\ 0.25,\ 0.5,\ 0.75$$

Linear stability scalings

Instability onset : $\mathrm{Ra}_Q^* = \widetilde{\mathrm{Ra}_Q^*} E^{-4/3}$ with $\widetilde{\mathrm{Ra}_Q^*} \approx 15$



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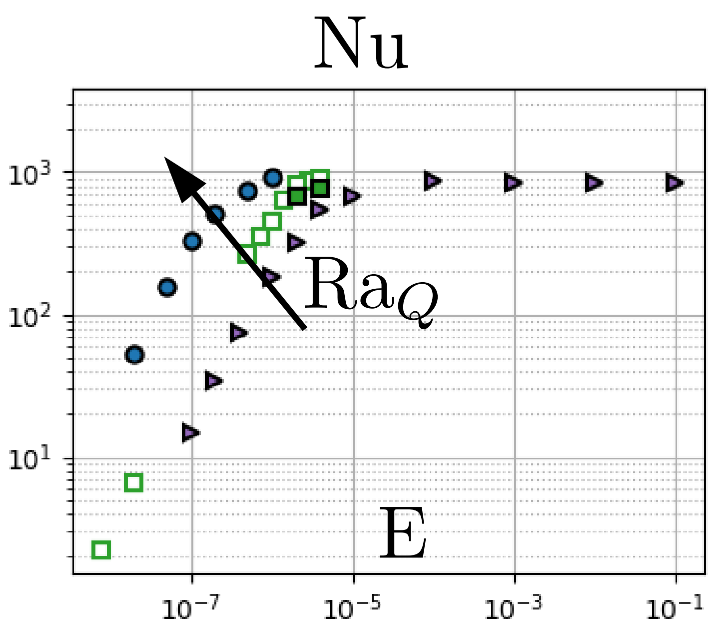
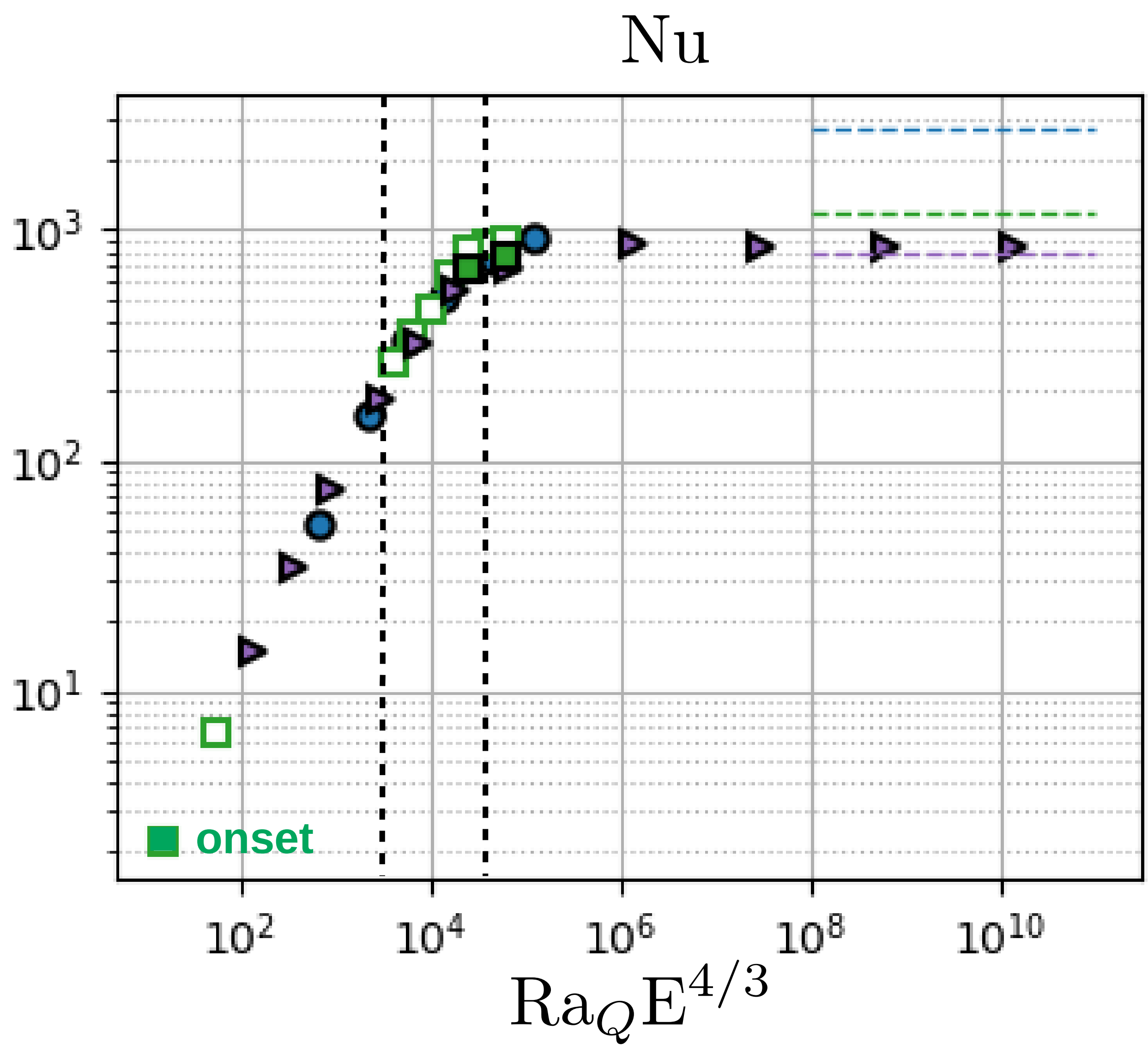
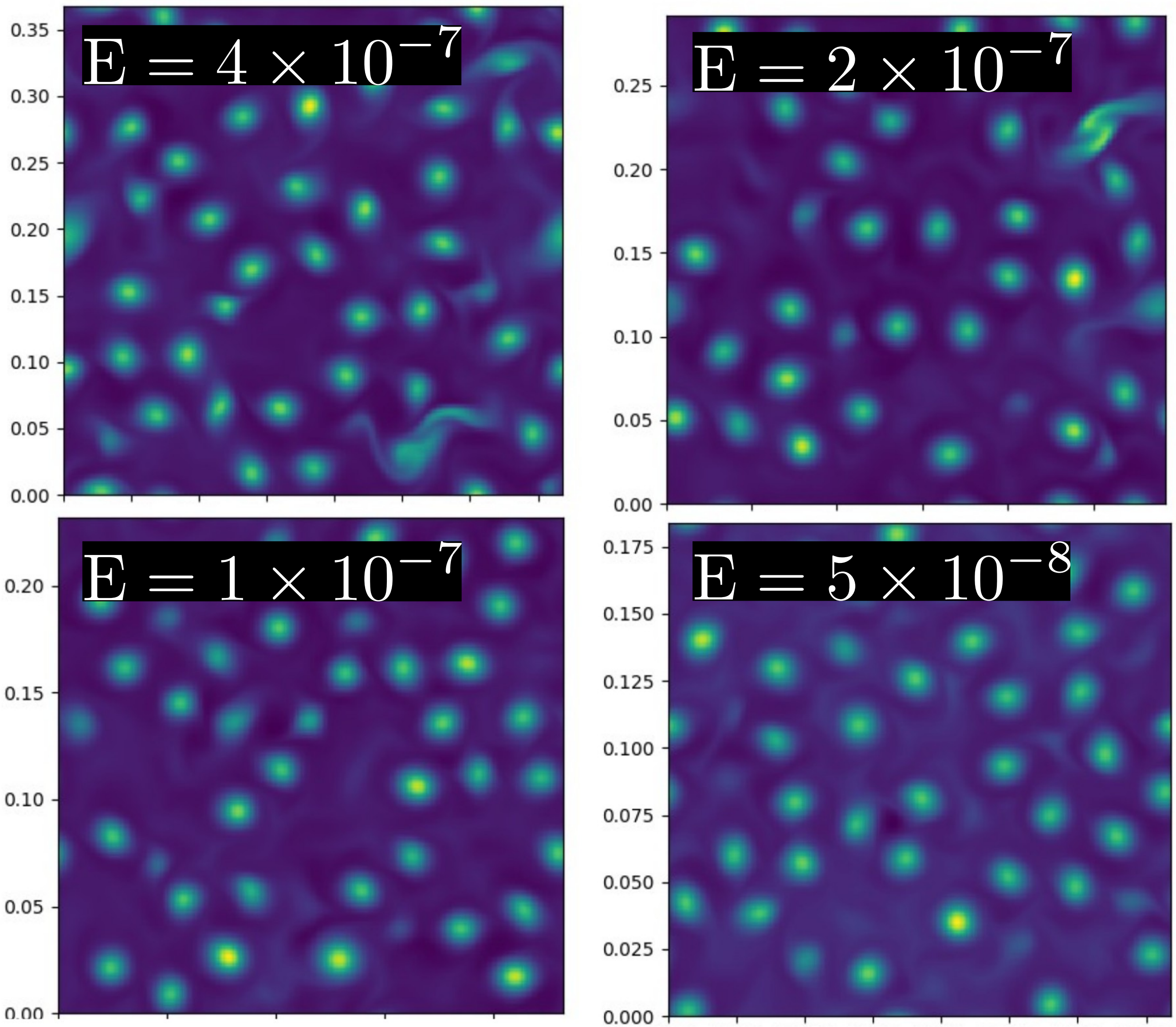




Linear stability scalings

Wavelength at onset : $L_\perp \sim \mathrm{E}^{1/3} H$

Timescale : $\tau \sim \mathrm{E}^{2/3} \left(H^2/\kappa\right)$



Slices : $\Theta(z = 0.25)$

Fixed $\widetilde{\mathrm{Ra}}_Q = 3000$

– Scale well captured

– The rescaled solutions look similar : **convergence towards an asymptotic solution ?**

Beyond linear stability scalings

Small parameter : $\varepsilon = \mathrm{E}^{1/3} \ll 1$

Anisotropic fields : $\nabla^2 = \varepsilon^{-2}\widetilde{\nabla}^2$ with $\widetilde{\nabla}^2 = \underbrace{\partial_{xx} + \partial_{yy}}_{\widetilde{\nabla_\perp}^2} + \varepsilon^2 \partial_{zz}$

Temperature : $\Theta = T(z,t) + \varepsilon\theta(x,y,z,t)$ [mean–fluctuation decomposition]

Advection : rescale $\boldsymbol{u} \to \varepsilon^{-1}\boldsymbol{u}$ so that $D_t^{\perp} = \partial_t + \boldsymbol{u}_\perp \cdot \boldsymbol{\nabla}_\perp + \varepsilon w \partial_z$

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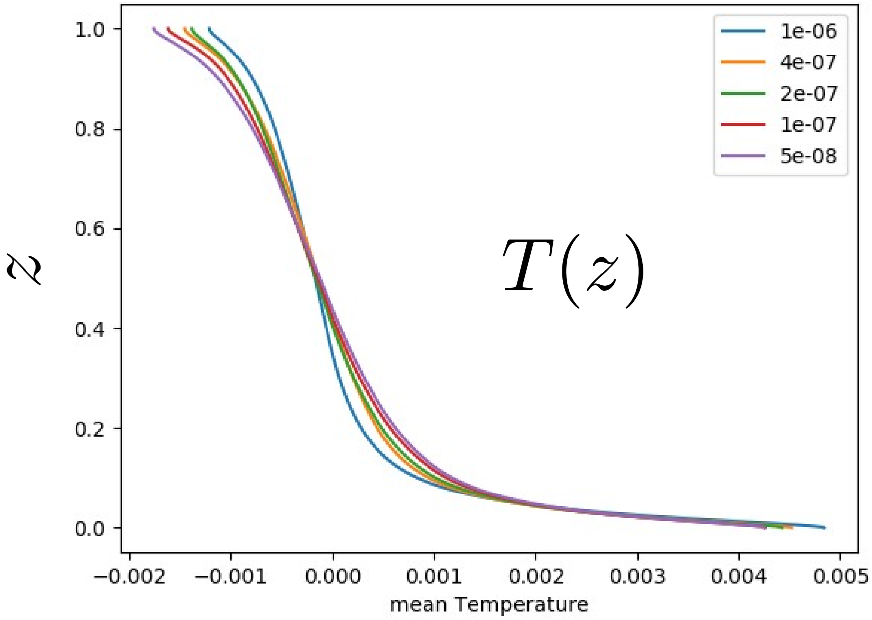
$$\frac{1}{\mathrm{Pr}} D_t u + \varepsilon^{-1}\hat{\boldsymbol{e}}_{\boldsymbol{z}} \times \boldsymbol{u} = \varepsilon^{-1}\boldsymbol{\nabla} p + \widetilde{\mathrm{Ra}_Q}\theta\hat{\boldsymbol{e}}_{\boldsymbol{z}} + \widetilde{\nabla}^2\boldsymbol{u}$$

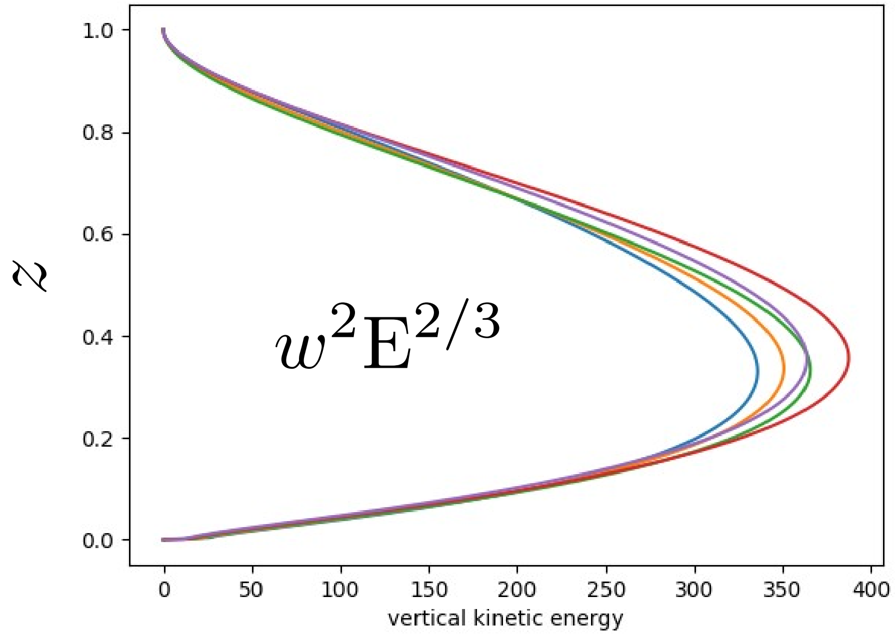
geostrophy

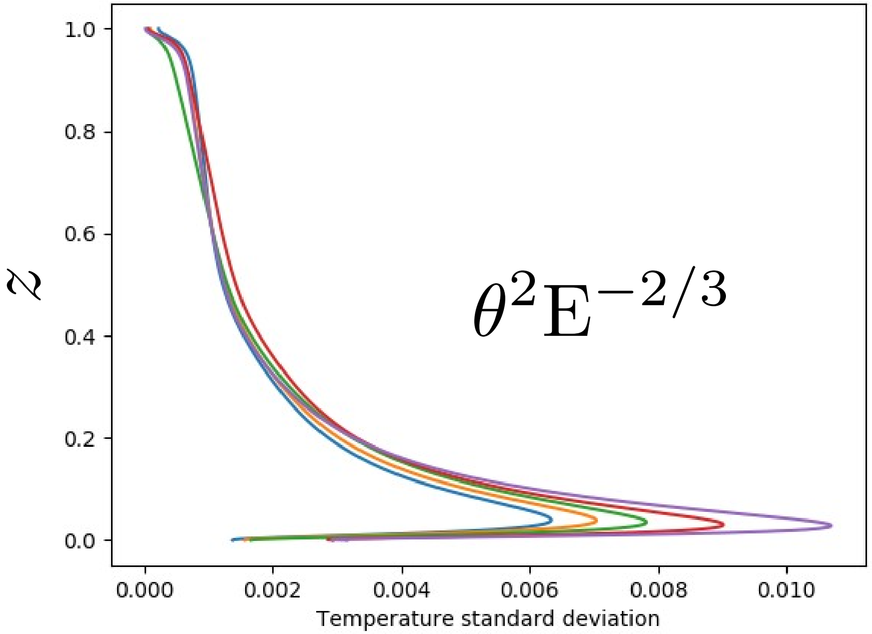
$$D_t\theta + \partial_z wT = \widetilde{\nabla}^2\theta$$

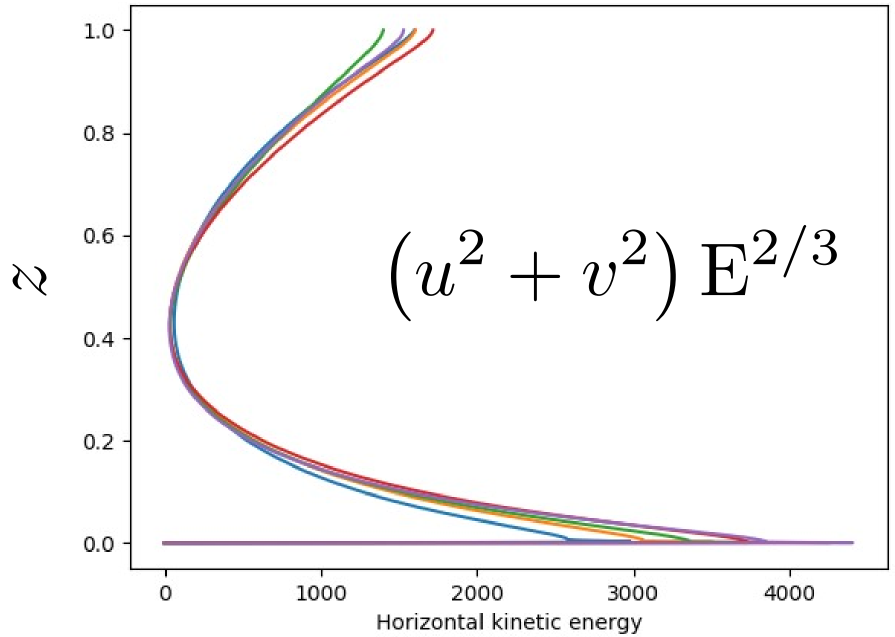
$$\varepsilon^{-2}D_tT + \partial_z\overline{w\theta}^{xy} = \partial_{zz}T + \left(\frac{1}{N_\ell}\exp\left(-z/\tilde{\ell}\right) - 1\right)$$

Convergence of the vertical profiles ; Fixed $\widetilde{\mathrm{Ra}_Q} = 3000$









Reduced equations : $\mathrm{E} \to 0$

Leading order geostrophy : $\boldsymbol{u} = \boldsymbol{\nabla} \times (\psi \cdot \hat{\boldsymbol{e}}_{\boldsymbol{z}}) + w \cdot \hat{\boldsymbol{e}}_{\boldsymbol{z}}$

Horizontal diffusion : $\widetilde{\nabla}^2_{\perp} = \partial_{xx} + \partial_{yy}$

Anisotropic advection : $D_t^{\perp} = \partial_t + \boldsymbol{u}_{\perp} \cdot \boldsymbol{\nabla}_{\perp}$

$$\frac{1}{\mathrm{Pr}} D_t^{\perp} \widetilde{\nabla}^2_{\perp} \psi + \partial_z w = \widetilde{\nabla}^4 \psi$$

$$\frac{1}{\mathrm{Pr}} D_t^{\perp} w = -\partial_z \psi + \widetilde{\mathrm{Ra}_Q} \theta + \widetilde{\nabla}^2_{\perp} w$$

$$D_t^{\perp} \theta + \partial_z w T = \widetilde{\nabla}^2 \theta$$

$$\partial_z \overline{w\theta}^{xy} = \partial_{zz} T + \left(\frac{1}{N_\ell} \exp\left(-z/\tilde{\ell}\right) - 1 \right)$$

Counterpart to reduced eqs for Rayleigh–Benard by Julien, Knobloch and Werne

Conclusion and Outlook

Rapidly rotating internally heated convection

— DNS solutions : convergence towards « master profiles » as $\mathrm{E} \to 0$
for kinetic energies, and mean temperature

— Temperature fluctuations : bottom boundary condition influence ?
(stress–free investigation in progress)

Can we leverage the rotation–induced scale separation and scalings ?
Solve the reduced equations

— Convergence of the DNS solutions towards the reduced solutions ?

— Heat flux prediction $\mathrm{Nu} = f(\widetilde{\mathrm{Ra}}_Q, \mathrm{Pr})$
Turbulent regimes ?